

Microlocal Analysis of an Ultrasound Transform with Circular Source and Receiver Trajectories

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This article is dedicated to Sigurdur Helgason on the occasion of his eighty fifth birthday. We thank him for creating so much beautiful mathematics and for being a friend and mentor to so many people in the field.

ABSTRACT. We consider a generalized Radon transform that is used in ultrasound reflection tomography. In our model, the ultrasound emitter and receiver move at a constant distance apart along a circle. We analyze the microlocal properties of the transform \mathcal{R} that arises from this model. As a consequence, we show that, for distributions with support contained in a disc D_b sufficiently inside the circle, $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator. We provide a local filtered back projection algorithm, $\mathcal{L} = \mathcal{R}^*D\mathcal{R}$ where D is a well-chosen differential operator. We prove that \mathcal{L} is an elliptic pseudodifferential operator of order 1 and so for $f \in \mathcal{E}'(D_b)$, $\mathcal{L}f$ shows all singularities of f , and we provide reconstructions illustrating this point. Finally, we discuss an extension with some modifications of our result outside of D_b .

1. Introduction

Ultrasound reflection tomography (URT) is one of the safest and most cost effective modern medical imaging modalities (e.g., see [13, 14, 15, 16] and the references there). During its scanning process, acoustic waves emitted from a source

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reflect from inhomogeneities inside the body, and their echoes are measured by a receiver. This measured data is then used to recover the unknown ultrasonic reflectivity function, which is used to generate cross-sectional images of the body.

In a typical setup of ultrasound tomography, the emitter and receiver are combined into one device (transducer). The transducer emits a short acoustic pulse into the medium, and then switches to receiving mode, recording echoes as a function of time. Assuming that the medium is weakly reflecting (i.e., neglecting multiple reflections), and that the speed of sound propagation c is constant¹, the echoes measured at time t uniquely determine the integrals of the reflectivity function over concentric spheres centered at the transducer location and radii $r = ct/2$ (see Fig. 1 (a) below, [16] and the references there). By focusing the transducer one can consider echoes coming only from a certain plane, hence measuring the integrals of the reflectivity function in that plane along circles centered at the transducer location [15]. Moving the transducer along a curve on the edge of the body, and repeating the measurements one obtains a two-dimensional family of integrals of the unknown function along circles. Hence the problem of image reconstruction in URT can be mathematically reduced to the problem of inverting a circular Radon transform, which integrates an unknown function of two variables along a two-dimensional family of circles.

In the case when the emitter and receiver are separated, the echoes recorded by a transducer correspond to the integrals of the reflectivity function along confocal ellipses. The foci of these ellipses correspond to the locations of the emitter and receiver moving along a fixed curve. While this more general setup has been gaining popularity in recent years (e.g., see [13, 14, 20]), the mathematical theory related to elliptical Radon transforms is relatively undeveloped.

In this paper we consider a setup where the separated emitter and receiver move along a circle at a fixed distance apart (see Fig. 1 (b)). The circular trajectory of their motion is both the simplest case mathematically and the one most often used in practice. By using a dilation and translation, we can assume the circle has radius $r = 1$ centered at 0. We study the microlocal properties of transform \mathcal{R} which integrates an unknown function along this family of ellipses.

We prove that \mathcal{R} is an elliptic Fourier integral operator (FIO) of order $-1/2$ using the microlocal framework of Guillemin and Guillemin-Sternberg [6, 8] for generalized Radon transforms. We use this to understand when the imaging operator $\mathcal{R}^*\mathcal{R}$ is a pseudodifferential operator. Specifically, we show that for distributions supported in a smaller disc (the disc D_b of (2.2)), a microlocal condition introduced by Guillemin [6], the so called Bolker assumption, is satisfied and, consequently, for such distributions $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator. We construct a differential operator D such that $\mathcal{L} = \mathcal{R}^*D\mathcal{R}$ is elliptic of order 1. From the tomographic point of view this means that using the measured data one can stably recover all singularities of objects supported inside that disc. We provide reconstructions that illustrate this. We note that, even when appropriately defined (see Remark 5.1), \mathcal{L} does not recover all singularities in the complement of D_b and \mathcal{L} can mask or add singularities. So, in this sense D_b is optimal. Stefanov and

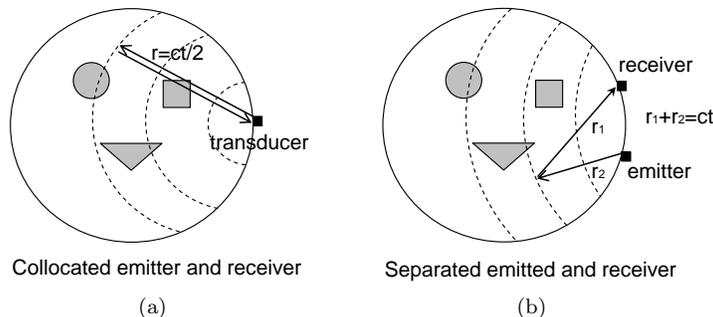


FIGURE 1. A sketch of integrating curves in URT

Uhlmann [18] show for a related problem in monostatic radar that singularities can be masked or added, even with arbitrary flight paths.

In Section 2, we introduce the basic notation and microlocal analysis as well as Guillemin's framework for understanding Radon transforms. In Section 3, we present the microlocal regularity theorem, and in Section 4 we present reconstructions from a local filtered backprojection algorithm (see equation (4.1)) that illustrates the conclusion of the main theorem. The proof of the microlocal regularity theorem is in Section 5.

2. Definitions and Preliminaries

We will first define the elliptical Radon transform we consider, provide the general framework for the microlocal analysis of this transform, and show that our transform fits within this framework.

2.1. The Elliptical Transform. Recall that in the URT model we consider in this paper, the emitter and receiver move along the circle of radius 1 centered at 0 and are at a fixed distance apart. We denote the fixed difference between the polar angles of emitter and receiver by 2α , where $\alpha \in (0, \pi/2)$ (see Fig. 2) and define

$$(2.1) \quad a = \sin \alpha, \quad b = \cos \alpha.$$

As we will see later our main result relies on the assumption that the support of the function is small enough. More precisely, we will assume our function is supported in the ball

$$(2.2) \quad D_b = \{x \in \mathbb{R}^2 \mid |x| < b\}.$$

We parameterize the trajectories of the transmitter (emitter) and receiver, respectively, as

$$\begin{aligned} \gamma_T(s) &= (\cos(s - \alpha), \sin(s - \alpha)) \\ \gamma_R(s) &= (\cos(s + \alpha), \sin(s + \alpha)) \quad \text{for } s \in [0, 2\pi]. \end{aligned}$$

¹This assumption is reasonable in ultrasound mammography, since the speed of sound is almost constant in soft tissue.

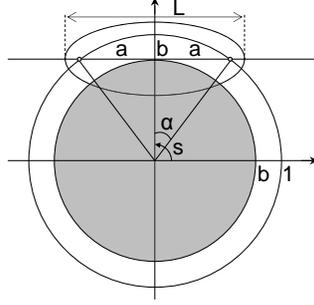


FIGURE 2. A sketch of the domain and the notations

Thus, the emitter and receiver rotate around the unit circle and are always $2a$ units apart. For $s \in [0, 2\pi]$ and $L > 2a$, let

$$E(s, L) = \{x \in \mathbb{R}^2 \mid |x - \gamma_T(s)| + |x - \gamma_R(s)| = L\}.$$

Note that the center of the ellipse $E(s, L)$ is $(b \cos s, b \sin s)$ and L is the diameter of the major axis of $E(s, L)$, the so called *major diameter*. This is why we require L to be greater than the distance between the foci, $2a$. As a function of s , the ellipse $E(s, L)$ is 2π -periodic, and so we will identify $s \in [0, 2\pi]$ with the point $(\cos s, \sin s)$ on the unit circle when convenient.

Let

$$Y = \{(s, L) \mid s \in [0, 2\pi], L > 2a\},$$

then Y is the set of parameters for the ellipses.

Let $(s, L) \in Y$. The elliptical Radon transform of a locally integrable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as

$$\mathcal{R}f(s, L) = \int_{x \in E(s, L)} f(x) dt(x)$$

where dt is the arc length measure on the ellipse $E(s, L)$. The backprojection transform is defined for $g \in C(Y)$ and $x \in D_b$ as

$$(2.3) \quad \mathcal{R}^*g(x) = \int_{s \in [0, 2\pi]} g(s, |x - \gamma_R(s)| + |x - \gamma_T(s)|) w(s, x) ds$$

where the positive smooth weight $w(s, x)$ is chosen so that \mathcal{R}^* is the L^2 adjoint of \mathcal{R} with measure dx on D_b and $ds dL$ on Y (see equation (2.13) in Example 2.2). Using the parameterization of ellipses (s, L) one sees that $\mathcal{R}^*g(x)$ integrates with a smooth measure over the set of all ellipses in our complex passing through $x \in D_b$. These transforms can be defined for distributions with support larger than D_b , but the definition of \mathcal{R}^* is more complicated for $x \notin D_b$ as will be discussed in Remark 5.1.

We can compose \mathcal{R} and \mathcal{R}^* on domain $\mathcal{E}'(D_b)$ for the following reasons. If $f \in \mathcal{D}(D_b)$ then $\mathcal{R}f$ has compact support in Y since $\mathcal{R}f(s, L)$ is zero for L near $2a$. Clearly, $\mathcal{R} : \mathcal{D}(D_b) \rightarrow \mathcal{D}(Y)$ is continuous so $\mathcal{R}^* : \mathcal{D}'(Y) \rightarrow \mathcal{D}'(D_b)$ is continuous. Since $x \in D_b$ in the definition of \mathcal{R}^* , (2.3), \mathcal{R}^* integrates over a compact set $[0, 2\pi]$. Therefore, $\mathcal{R}^* : \mathcal{E}(Y) \rightarrow \mathcal{E}(D_b)$ is continuous, so $\mathcal{R} : \mathcal{E}'(D_b) \rightarrow \mathcal{E}'(Y)$ is continuous. Therefore, \mathcal{R}^* can be composed with \mathcal{R} on domain $\mathcal{E}'(D_b)$.

2.2. Microlocal Definitions. We now introduce some notation so we can describe our operators microlocally. Let X and Y be smooth manifolds and let

$$\mathcal{C} \subset T^*(Y) \times T^*(X),$$

then we let

$$\mathcal{C}' = \{(y, \eta, x, \xi) \mid (y, \eta, x, -\xi) \in \mathcal{C}\}.$$

The transpose relation is $\mathcal{C}^t \subset T^*(X) \times T^*(Y)$:

$$\mathcal{C}^t = \{(x, \xi, y, \eta) \mid (y, \eta, x, \xi) \in \mathcal{C}\}$$

If $\mathcal{D} \subset T^*(X) \times T^*(Y)$, then the composition $\mathcal{D} \circ \mathcal{C}$ is defined

$$\begin{aligned} \mathcal{D} \circ \mathcal{C} &= \{(x', \xi', x, \xi) \mid \exists (y, \eta) \in T^*(Y) \\ &\text{with } (x', \xi', y, \eta) \in \mathcal{D}, (y, \eta, x, \xi) \in \mathcal{C}\}. \end{aligned}$$

2.3. The Radon Transform and Double Fibrations in General. Guillemin first put the Radon transform into a microlocal framework, and we now describe this approach and explain how our transform \mathcal{R} fits into this framework. We will use this approach to prove Theorem 3.1.

Guillemin used the ideas of pushforwards and pullbacks to define Radon transforms and show they are Fourier integral operators (FIOs) in the technical report [5], and these ideas were outlined in [8, pp. 336-337, 364-365] and summarized in [6]. He used these ideas to define FIOs in general in [7, 8]. The dependence on the measures and details of the proofs for the case of equal dimensions were given in [17].

Given smooth connected manifolds X and Y of the same dimension, let $Z \subset Y \times X$ be a smooth connected submanifold of codimension $k < \dim(X)$. We assume that the natural projections

$$(2.4) \quad \begin{array}{ccc} & Z & \\ \pi_L \swarrow & & \searrow \pi_R \\ Y & & X \end{array}$$

are both fiber maps. In this case, we call (2.4) a *double fibration*. This framework was used by Helgason [9] to define Radon transforms in a group setting, and it was later generalized to manifolds without a group structure [4].

Following Guillemin and Sternberg, we assume that π_R is a *proper map*; that is, the fibers of $\pi_R : Z \rightarrow X$ are compact.

The double fibration allows us to define sets of integration for the Radon transform and its dual as follows. For each $y \in Y$ let

$$E(y) = \pi_R \left(\pi_L^{-1}(\{y\}) \right),$$

then $E(y)$ is a subset of X that is diffeomorphic to the fiber of $\pi_L : Z \rightarrow Y$. For each $x \in X$ let

$$F(x) = \pi_L \left(\pi_R^{-1}(\{x\}) \right),$$

then $F(x) \subset Y$ is diffeomorphic to the fiber of $\pi_R : Z \rightarrow X$. Since π_R is proper, $F(x)$ is compact.

Guillemin defined the Radon transform and its dual using pushforwards and pullbacks. The pullback of the function $f \in C^\infty(X)$ is

$$\pi_R^* f(z) = f(\pi_R(z))$$

and the pushforward to X of a measure ν on Z is the measure satisfying

$$\int_X f(x) d\pi_{R*}(\nu) = \int_Z (\pi_R^* f) d\nu.$$

The pushforward and pullback for π_L are defined similarly.

By choosing smooth nowhere zero measures μ on Z , m on X , and n on Y , one defines the generalized Radon transform of $f \in C_c^\infty(X)$ as the function Rf for which

$$(2.5) \quad (Rf)n = \pi_{L*}((\pi_R^* f)\mu).$$

The dual transform for $g \in C^\infty(Y)$ is the function R^*g for which

$$(R^*g)m = \pi_{R*}((\pi_L^* g)\mu).$$

This definition is natural because R^* is automatically the dual to R by the duality between pushforwards and pullbacks.

The measures μ , n and m give the measures of integration for R and R^* as follows. Since $\pi_L : Z \rightarrow Y$ is a fiber map locally above $y \in Y$, the measure μ can be written as a product of the measure n and a smooth measure on the fiber. This fiber is diffeomorphic to $E(y)$, and the measure on the fiber can be pushed forward using this diffeomorphism to a measure μ_y on $E(y)$: the measure μ_y satisfies $\mu = \mu_y \times n$ (under the identification of $E(y)$ with the fiber of Z above y), and the generalized Radon transform defined by (2.5) can be written

$$Rf(y) = \int_{x \in E(y)} f(x) d\mu_y(x)$$

In a similar way, the measure μ_x on each set $F(x)$ satisfies $\mu = \mu_x \times m$ (under the identification of the fiber of π_R with $F(x)$), and the dual transform can be written

$$R^*g(x) = \int_{y \in F(x)} g(y) d\mu_x(y)$$

[8] (see also [17, p. 333]).

Since the sets $F(x)$ are compact, one can compose R^* and R for $f \in C_c(X)$. We include the uniqueness assumptions $E(y_1) = E(y_2)$ if and only if $y_1 = y_2$ and $F(x_1) = F(x_2)$ if and only if $x_1 = x_2$.

Guillemin showed ([5, 6] and with Sternberg [8]) that R is a Fourier integral distribution associated with integration over Z and canonical relation $\mathcal{C} = (N^*(Z) \setminus \{\mathbf{0}\})'$. To understand the properties of R^*R , one must investigate the mapping properties of \mathcal{C} . Let $\Pi_L : \mathcal{C} \rightarrow T^*(Y)$ and $\Pi_R : \mathcal{C} \rightarrow T^*(X)$ be the projections. Then we have the following diagram:

$$(2.6) \quad \begin{array}{ccc} & \mathcal{C} & \\ \Pi_L \swarrow & & \searrow \Pi_R \\ T^*(Y) & & T^*(X) \end{array}$$

This diagram is the microlocal version of (2.4).

DEFINITION 2.1 ([5, 6]). Let X and Y be manifolds with $\dim(Y) = \dim(X)$ and let $\mathcal{C} \subset (T^*(Y) \times T^*(X)) \setminus \{\mathbf{0}\}$ be a canonical relation. Then, \mathcal{C} satisfies the *Bolker Assumption* if

$$\Pi_Y : \mathcal{C} \rightarrow T^*(Y)$$

is an injective immersion.

This definition was originally proposed by Guillemin [5],[6, p. 152], [8, p. 364-365] because Ethan Bolker proved R^*R is injective under a similar assumption for a finite Radon transform. Guillemin proved that if the measures that define the Radon transform are smooth and nowhere zero, and if the Bolker Assumption holds (and R is defined by a double fibration for which π_R is proper), then R^*R is an elliptic pseudodifferential operator.

Since we assume $\dim(Y) = \dim(X)$, if $\Pi_Y : \mathcal{C} \rightarrow T^*(Y)$ is an injective immersion, then Π_Y maps to $T^*(Y) \setminus \{\mathbf{0}\}$ and Π_X is also an immersion [10]. Therefore, Π_X maps to $T^*(X) \setminus \{\mathbf{0}\}$. So, under the Bolker Assumption, $\mathcal{C} \subset (T^*(Y) \setminus \{\mathbf{0}\}) \times (T^*(X) \setminus \{\mathbf{0}\})$ and so R is a Fourier integral operator according to the definition in [19].

We now put our elliptical transform into this framework.

EXAMPLE 2.2. For our transform \mathcal{R} , the *incidence relation* is

$$(2.7) \quad Z = \{(s, L, x) \in Y \times D_b \mid x \in E(s, L)\}.$$

The double fibration is

$$(2.8) \quad \begin{array}{ccc} & Z & \\ \pi_L \swarrow & & \searrow \pi_R \\ Y & & D_b \end{array}$$

and both projections are fiber maps. These projections define the sets we integrate over: the ellipse $E(s, L) = \pi_R(\pi_L^{-1}(\{(s, L)\}))$ and the closed curve in Y

$$F(x) = \pi_L(\pi_R^{-1}(\{x\})) = \{(s, \ell(s, x)) \mid s \in [0, 2\pi]\}$$

where

$$(2.9) \quad \ell(s, x) = |x - \gamma_R(s)| + |x - \gamma_T(s)|.$$

Note that π_R is proper and $F(x)$ is diffeomorphic to the circle.

One chooses measure $m = dx$ on D_b and measure $n = ds dL$ on Y . For each $(s, L) \in Y$ one parameterizes the ellipse $E(s, L) \cap D_b$ by arc length with coordinate t so that

$$(2.10) \quad x = x(s, L, t) \in E(s, L)$$

is a smooth function of (s, L, t) . Then, Z can be parameterized by (s, L, t) and this gives the measure we use on Z , $\mu = ds dL dt$. Since the measure on Y is $ds dL$ and $\mu = (ds dL) dt$, the measure on the fiber of π_L is dt . This gives measure $\mu_{(s, L)} = dt$ which is the arc length measure on the ellipse $E(s, L)$.

To find the measure on $F(x)$ note that the factor, $w(s, x)$, giving this measure satisfies

$$(2.11) \quad ds dL dt = w(s, x) ds dx, \quad \text{or} \quad dL dt = w(s, x) dx.$$

For fixed s , $(L, t) \mapsto x(s, L, t)$ give coordinates on D_b . The Jacobian factor $w(s, x)$ in equation (2.11) must be

$$(2.12) \quad w(s, x) = |\partial_x \ell| |\partial_x t|$$

where $L = \ell(s, x)$ and t are considered as functions of x and where ∂_x is the gradient in x and ∂_t is the derivative in t . This expression is valid since the vectors in (2.12) are perpendicular because the first vector is normal to the ellipse $E(s, L)$ at $x(s, L, t)$ and the second vector is tangent to the ellipse. Since t parameterizes

arc length, the second factor on the right-hand side of (2.12) is 1. This means $w(s, x) = |\partial_x \ell(s, x)|$. To calculate this expression for $w(s, x)$ we note that

$$\partial_x \ell(s, x) = \frac{x - \gamma_R(s)}{|x - \gamma_R(s)|} + \frac{x - \gamma_T(s)}{|x - \gamma_T(s)|}.$$

Since this expression is the sum of two unit vectors, its length is $2 \cos(\varphi/2)$ where φ is the angle between these two vectors. A calculation shows that this is

$$(2.13) \quad w(s, x) = 2 \cos(\varphi/2) = \sqrt{2 + 2 \left(\frac{(x - \gamma_R(s)) \cdot (x - \gamma_T(s))}{|x - \gamma_R(s)| |x - \gamma_T(s)|} \right)}$$

where $\cos \varphi$ is the expression in parentheses in the square root. Note that $\varphi < \pi$ since x is not on the segment between the two foci. Therefore, the weight $w(s, x) \neq 0$. The second expression (found using the law of cosines) gives w explicitly in terms of s and x .

This discussion shows that \mathcal{R} and \mathcal{R}^* satisfy the conditions outlined in the first part of this section so that Guillemin and Sternberg's framework can be applied.

3. The Main Result

We now state the main result of this article. Proofs are in Section 5.

THEOREM 3.1. *Let $\alpha \in (0, \pi/2)$ be a constant and let*

$$\begin{aligned} \gamma_T(s) &= (\cos(s - \alpha), \sin(s - \alpha)) \quad \text{and} \\ \gamma_R(s) &= (\cos(s + \alpha), \sin(s + \alpha)) \quad \text{for } s \in [0, 2\pi] \end{aligned}$$

be the trajectories of the ultrasound emitter and receiver respectively. Denote by $\mathcal{E}'(D_b)$ the space of distributions supported in the open disc, D_b , of radius b centered at 0, where $b = \cos \alpha$.

The elliptical Radon transform \mathcal{R} when restricted to the domain $\mathcal{E}'(D_b)$ is an elliptic Fourier integral operator (FIO) of order $-1/2$. Let $\mathcal{C} \subset T^(Y) \times T^*(D_b)$ be the canonical relation associated to \mathcal{R} . Then, \mathcal{C} satisfies the Bolker Assumption (Definition 2.1).*

As a consequence of this result, we have the following corollary.

COROLLARY 3.2. *The composition of \mathcal{R} with its L^2 adjoint \mathcal{R}^* when restricted as a transformation from $\mathcal{E}'(D_b)$ to $\mathcal{D}'(D_b)$ is an elliptic pseudo-differential operator of order -1 .*

This corollary shows that, for $\text{supp } f \subset D_b$, the singularities of $\mathcal{R}^* \mathcal{R} f$ (as a distribution on D_b) are at the same locations and co-directions as the singularities of f , that is, the wavefront sets are the same. In other words, $\mathcal{R}^* \mathcal{R}$ reconstructs all the singularities of f . In the next section, we will show reconstructions from an algorithm.

REMARK 3.3. Our theorems are valid for any elliptic FIO that has the canonical relation \mathcal{C} given by (5.1) because the composition calculus of FIO is determined by the canonical relation. This means that our theorems would be true for any other model of this bistatic ultrasound problem having the same canonical relation \mathcal{C} .

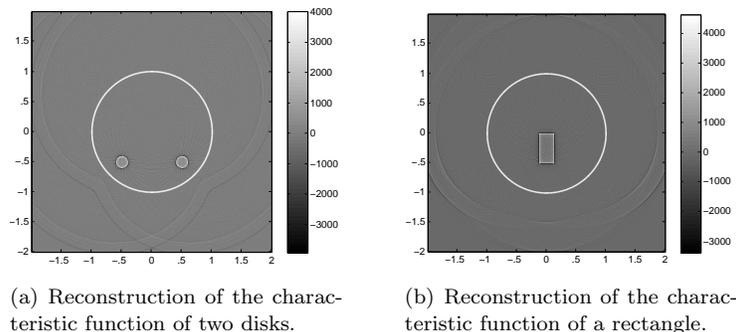


FIGURE 3. Reconstructions using the operator \mathcal{L} in (4.1) by Tufts Senior Honors Thesis student Howard Levinson [12]. The reconstructions were done with 300 values of L and 360 values of s , and $\alpha = \pi/32$.

4. Reconstructions from a Local Backprojection Algorithm

In this section, we describe a local backprojection type algorithm and show reconstructions from simulated data. The reconstructions and algorithm development were a part of an REU project and senior honors thesis [12] of Tufts University undergraduate Howard Levinson. Prof. Quinto's algorithm

$$(4.1) \quad \mathcal{L}f = \mathcal{R}^*(-\partial^2/\partial L^2)\mathcal{R}f$$

is a generalization of Lambda Tomography [3, 2], which is a filtered backprojection type algorithm with a derivative filter. Note that the algorithm is local in the sense that one needs only data over ellipses near a point $x \in D_b$ to reconstruct $\mathcal{L}(f)(x)$. We infer from our next theorem that \mathcal{L} detects all singularities inside D_b .

THEOREM 4.1. *The operator $\mathcal{L} : \mathcal{E}'(D_b) \rightarrow \mathcal{D}'(D_b)$ is an elliptic pseudodifferential operator of order 1.*

PROOF. The order of \mathcal{L} is one because \mathcal{R} and \mathcal{R}^* are both of order $-1/2$ and $-\partial^2/\partial L^2$ is of order 2. \mathcal{L} is elliptic for the following reasons. From Theorem 3.1, we know that \mathcal{R} is elliptic for distributions in $\mathcal{E}'(D_b)$. Then, $-\partial^2/\partial L^2$ is elliptic on distributions with wavefront in $\Pi_L(\mathcal{C})$ because the dL component of such distributions is never zero as can be seen from (5.1). Finally, by the Bolker Assumption on \mathcal{C} , one can compose \mathcal{R}^* and $-\partial^2/\partial L^2\mathcal{R}$ to get an elliptic pseudodifferential operator for distributions supported on D_b . \square

Mr. Levinson also tried replacing $-\partial^2/\partial L^2$ by $-\partial^2/\partial s^2$ in (4.1) but some boundaries were not as well-defined in the reconstructions. This reflects the fact that the analogous operator corresponding to \mathcal{L} is not elliptic since the symbol of $-\partial^2/\partial s^2$ is zero on a subset of $\Pi_L(\mathcal{C})$. For instance, the ds component is 0 on covectors corresponding to points on the minor axis of the ellipse $E(s, L)$ determined by s and L .

REMARK 4.2. These reconstructions are consistent with Theorem 4.1 since all singularities of the objects are visible in the reconstructions and no singularities are

added inside D_b . Notice that there are added singularities in the reconstructions in Figure 3, but they are outside D_b . Added singularities are to be expected because of the left-right ambiguity: an object on one side of the major axis of an ellipse has the same integral over that ellipse as its mirror image in the major axis. This is most pronounced in the common-offset case in which the foci γ_T and γ_R travel on a line [11].

5. Proofs of Theorem 3.1 and Corollary 3.2

If g is a function of $(s, x) \in [0, 2\pi] \times D_b$, then we will let $\partial_s g$ denote the first derivative of g with respect to s , and $\partial_x g$ will denote the derivative of g with respect to x . When $x = (x_1, x_2) \in \mathbb{R}^2$, we define $\partial_x g d\mathbf{x} = \frac{\partial g}{\partial x_1} d\mathbf{x}_1 + \frac{\partial g}{\partial x_2} d\mathbf{x}_2$. Here we use boldface for covectors, such as $d\mathbf{x}$, $d\mathbf{s}$, and $d\mathbf{L}$ to distinguish them from measures, such as dx , ds , and dL .

PROOF OF THEOREM 3.1. First, we will calculate $\mathcal{C} = (N^*(Z) \setminus \{\mathbf{0}\})'$ where Z is given by (2.7) and then show that \mathcal{C} satisfies the Bolker Assumption. The set Z is defined by $L - \ell(s, x) = 0$ where ℓ is defined by (2.9) and the differential of this function is a basis for $N^*(Z)$. Therefore, $\mathcal{C} = N^*(Z) \setminus \{\mathbf{0}\}'$ is given by

$$(5.1) \quad \mathcal{C} = \{(s, L, -\omega \partial_s \ell d\mathbf{s} + \omega d\mathbf{L}, x, \omega \partial_x \ell d\mathbf{x}) \mid (s, L) \in Y, x \in E(s, L), \omega \neq 0\}.$$

The Schwartz kernel of \mathcal{R} is integration on Z (e.g., [17, Proposition 1.1]) and so \mathcal{R} is a Fourier integral distribution associated to \mathcal{C} [6].

We now show that the projection

$$(5.2) \quad \begin{aligned} \Pi_L(s, L, -\omega \partial_s \ell(s, x) d\mathbf{s} + \omega d\mathbf{L}, x, \omega \partial_x \ell(s, x) d\mathbf{x}) \\ = (s, L, -\omega \partial_s \ell(s, x) d\mathbf{s} + \omega d\mathbf{L}) \end{aligned}$$

is an injective immersion. Let (s, L, η_s, η_L) be coordinates on $T^*(Y)$. Note that s, L and $\omega = \eta_L$ are determined by Π_L , so we just need to determine $x \in D_b$ from (5.2). From the value of L in (5.2) we know that $x \in E(s, L) \cap D_b$, so we fix L . By rotation invariance, we can assume $s = 0$. Now, we let

$$(5.3) \quad E_b = E(0, L) \cap D_b.$$

Let m be the length of the curve E_b and let $x(t)$ be a parameterization of E_b by arc length for $t \in (0, m)$ so that $x(t)$ moves up E_b (x_2 increases) as t increases.

The η_s coordinate in (5.2) with $x = x(t)$ and $\omega = -1$ is

$$(5.4) \quad \eta_s(t) = \partial_s \ell(s, x) = \frac{x(t) - \gamma_R(0)}{|x(t) - \gamma_R(0)|} \cdot \gamma'_R(0) + \frac{x(t) - \gamma_T(0)}{|x(t) - \gamma_T(0)|} \cdot \gamma'_T(0).$$

To show Π_L is an injective immersion, we show that $\eta_s(t)$ has a positive derivative everywhere on $(0, m)$. To do this, we consider the terms in (5.4) separately.

The first term

$$(5.5) \quad T_1(t) = \frac{(x(t) - \gamma_R(0))}{|x(t) - \gamma_R(0)|} \cdot \gamma'_R(0)$$

is the cosine of the angle, $\beta_1(t)$, between the vector $(x(t) - \gamma_R(0))$ and the tangent vector $\gamma'_R(0)$:

$$T_1(t) = \cos(\beta_1(t)).$$

The vector $x(t) - \gamma_R(0)$ is transversal to the ellipse $E(s, L)$ at $x(t)$ since $\gamma_R(0)$ is inside the ellipse and $x(t)$ is on the ellipse. Therefore, $\beta'_1(t) \neq 0$ for all $t \in (0, m)$. Since $x(t)$ is inside the unit disk, $\beta_1(t)$ is neither 0 nor π so $T_1(t) = \cos \beta_1(t)$ is

neither maximum or minimum. This implies that $T_1'(t) \neq 0$ for all $t \in (0, m)$. By the Intermediate Value Theorem T_1' must be either positive or negative everywhere on $(0, m)$. Since $x(t)$ travels up E_b as t increases, $T_1(t) = \cos(\beta_1(t))$ increases, and so $T_1'(t) > 0$ for all $t \in (0, m)$. A similar argument shows that the second term in (5.4) has positive derivative for $t \in (0, m)$. Therefore, $\partial_t \eta_s(t) > 0$ for all $t \in (0, m)$ and the Inverse Function Theorem shows that the function $\eta_s(t)$ is invertible by a smooth function. This proves that Π_L is an injective immersion.

As mentioned after Definition 2.1, the projections Π_L and Π_R map away from the $\mathbf{0}$ section. Therefore, \mathcal{R} is a Fourier integral operator [19]. Since the measures μ , dx and $ds dL$ are nowhere zero, \mathcal{R} is elliptic. The order of \mathcal{R} is given by $(\dim(Y) - \dim(Z))/2$ (see e.g., [6, Theorem 1] which gives the order of $\mathcal{R}^*\mathcal{R}$). In our case, Z has dimension 3 and Y has dimension 2, hence \mathcal{R} has order $-1/2$. This concludes the proof of Theorem 3.1. \square

PROOF OF COROLLARY 3.2. The proof that $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator follows from Guillemin's result [6, Theorem 1] as a consequence of Theorem 3.1 and the fact $\pi_R : Z \rightarrow \mathbb{R}^2$ is proper. We will outline the proof since the proof for our transform is simple and instructive. As discussed previously, we can compose \mathcal{R}^* and \mathcal{R} for distributions in $\mathcal{E}'(D_b)$.

By Theorem 3.1, \mathcal{R} is an elliptic Fourier integral operator associated with \mathcal{C} . By the standard calculus of FIO, \mathcal{R}^* is an elliptic FIO associated to \mathcal{C}^t . Because the Bolker Assumption holds above D_b , \mathcal{C} is a local canonical graph and so the composition $\mathcal{R}^*\mathcal{R}$ is a FIO for functions supported in D_b . Now, because of the injectivity of Π_Y , $\mathcal{C}^t \circ \mathcal{C} \subset \Delta$ where Δ is the diagonal in $(T^*(D_b) \setminus \{\mathbf{0}\})^2$ by the clean composition of Fourier integral operators [1].

To show $\mathcal{C}^t \circ \mathcal{C} = \Delta$, we need to show $\Pi_R : \mathcal{C} \rightarrow T^*(D_b) \setminus \{\mathbf{0}\}$ is surjective. This will follow from (5.1) and a geometric argument. Let $(x, \xi) \in T^*(D_b) \setminus \{\mathbf{0}\}$. We now prove there is a $(s, L) \in Y$ such that (x, ξ) is conormal the ellipse $E(s, L)$. First note that any ellipse $E(s, L)$ that contains x must have $L = |x - \gamma_R(s)| + |x - \gamma_T(s)|$. As s ranges from 0 to 2π the normal line at x to the ellipse $E(s, |x - \gamma_R(s)| + |x - \gamma_T(s)|)$ at s rotates completely around 2π radians and therefore for some value of $s_0 \in [0, 2\pi]$, (x, ξ) must be conormal $E(s_0, |x - \gamma_R(s_0)| + |x - \gamma_T(s_0)|)$. Since the ellipse is given by the equation $L = |x - \gamma_R(s)| + |x - \gamma_T(s)|$, its gradient is normal to the ellipse at x ; conormals co-parallel this gradient are exactly of the form

$$\xi = \omega \left(\frac{x - \gamma_R(s_0)}{|x - \gamma_R(s_0)|} + \frac{x - \gamma_T(s_0)}{|x - \gamma_T(s_0)|} \right) \mathbf{dx}$$

for some $\omega \neq 0$. Using (5.1), we see that for this s_0 , x , ω and $L = |x - \gamma_R(s_0)| + |x - \gamma_T(s_0)|$, there is a $\lambda \in \mathcal{C}$ with $\Pi_R(\lambda) = (x, \xi)$. This finishes the proof that Π_R is surjective. Note that one can also prove this using the fact that π_R is a fibration (and so a submersion) and a proper map, but our proof is elementary. This shows that $\mathcal{R}^*\mathcal{R}$ is an elliptic pseudodifferential operator viewed as an operator from $\mathcal{E}'(D_b) \rightarrow \mathcal{D}'(D_b)$. Because the \mathcal{R}^* and \mathcal{R} have order $-1/2$, $\mathcal{R}^*\mathcal{R}$ has order -1 . \square

REMARK 5.1. In this remark, we investigate the extent to which our results can be extended to the open unit disk, D_1 .

The Guillemin framework discussed in Section 2.3 breaks down outside D_b because $\pi_R : Z \rightarrow D_1$ is no longer a proper map. If $x \in D_1 \setminus \text{cl}(D_b)$, then there are two degenerate ellipses through x : there are two values of s such that $\ell(s, x) = 2a$, so, for these values of s , the ‘‘ellipse’’ $E(s, \ell(s, x))$ is the segment between the foci

$\gamma_R(s)$ and $\gamma_T(s)$, and such points $(s, 2a)$ are not in Y . This means that the fibers of π_R are not compact above such points. This is more than a formal problem since it implies we cannot evaluate \mathcal{R}^* on arbitrary distributions in $\mathcal{D}'(Y)$. Basically, one cannot integrate arbitrary distributions on Y over this noncompact fiber of π_R . More formally, because $\mathcal{R} : \mathcal{D}(D_1) \rightarrow \mathcal{E}(Y)$, $\mathcal{R}^* : \mathcal{E}'(Y) \rightarrow \mathcal{D}'(D_1)$. Similarly, $\mathcal{R} : \mathcal{E}'(D_1) \rightarrow \mathcal{D}'(Y)$. Therefore, one cannot use standard arguments to compose \mathcal{R} and \mathcal{R}^* for $f \in \mathcal{E}'(D_1)$ without using a cutoff function, φ , on Y that is zero near $L = 2a$. With such a cutoff function $\mathcal{R}^*\varphi\mathcal{R}$ can be defined on domain $\mathcal{E}'(D_1)$.

Now we consider the operator $\mathcal{R}^*\varphi\mathcal{R}$ on domain $\mathcal{E}'(D_1)$. It is straightforward to see that Π_L is not injective if points outside of D_b are included. For each ellipse $E(s, L)$, covectors in \mathcal{C} above the two vertices of $E(s, L)$ on its minor axis project to the same covector under Π_L . This is even true for thin ellipses for which both halves meet D_1 . This means that, even using a cutoff function φ on Y , $\mathcal{R}^*\varphi\mathcal{R}$ will not be a pseudodifferential operator (unless φ is zero for all L for which both halves of $E(s, L)$ intersect D_1). Let \mathcal{C} now denote the canonical relation of \mathcal{R} over D_1 . Because Π_L is not injective, $\mathcal{C}^t \circ \mathcal{C}$ contains covectors not on the diagonal and, therefore, $\mathcal{R}^*\varphi\mathcal{R}$ is not a pseudodifferential operator.

For these reasons, we now introduce a *half-ellipse transform*. Let the curve, $E_h(s, L)$ be the half of the ellipse $E(s, L)$ that is on the one side of the line between the foci $\gamma_R(s)$ and $\gamma_T(s)$ closer to the origin. This half of the ellipse $E(s, L)$ meets D_b and the other half does not. We denote the transform that integrates functions on D_1 over these half-ellipses by \mathcal{R}_h and its dual by \mathcal{R}_h^* . The incidence relation of \mathcal{R}_h will be denoted \mathcal{C}_h and its canonical relation will be \mathcal{C}_h .

We now use arguments from the proof of Theorem 3.1 to show $\Pi_L : \mathcal{C}_h \rightarrow T^*(Y)$ is an injective immersion. The parameterization $x(t)$ of E_b below equation (5.3) can be extended for t in a larger interval $(\alpha, \beta) \supset (0, m)$ to become a parameterization of $E_h(0, L) \cap D_1$. The function T_1 in equation (5.5) is defined for $t \in (\alpha, \beta)$, and the proof we gave that $T_1'(t) > 0$ is valid for such points since they are inside the unit disk (see the last paragraph of the proof of Theorem 3.1). For a similar reason, the second term in (5.4) has a positive derivative for $t \in (\alpha, \beta)$. As with our proof for D_b , this shows that $\Pi_L : \mathcal{C}_h \rightarrow Y$ is an injective immersion.

We now investigate generalizations of Corollary 3.2. For the same reasons as for \mathcal{R} , we cannot compose \mathcal{R}_h^* and \mathcal{R}_h for distributions on D_1 . We choose a smooth cutoff function $\varphi(L)$ that is zero near $L = 2a$ and equal to 1 for $L > 2a + \varepsilon$ for some small $\varepsilon > 0$. Then, $\mathcal{R}_h^*\varphi\mathcal{R}_h$ is well defined on $\mathcal{E}'(D_1)$. Because Π_L satisfies the Bolker Assumption, $\mathcal{R}_h^*\varphi\mathcal{R}_h$ is a pseudodifferential operator.

We will now outline a proof that $\mathcal{R}_h^*\varphi\mathcal{R}_h$ is elliptic on D_1 if ε is small enough. Let $x \in D_1$. Then, one can use the same normal line argument as in the last paragraph of the proof of Corollary 3.2 to show that Π_R is surjective. Namely, for each $x \in D_1$ and each conormal, ξ above x , there are two ellipses containing x and conormal to ξ , and at least one of them will meet x in the inner half. That is, for some $s \in [0, 2\pi)$, $E_h(s, \ell(s, x))$ is conormal to (x, ξ) . Furthermore, if ε is chosen small enough (independent of (x, ξ)), $\ell(s, x)$ will be greater than $2a + \varepsilon$. So $\mathcal{R}_h^*\varphi\mathcal{R}_h$ is an elliptic pseudodifferential operator.

Finally, note that $\Pi_R : \mathcal{C}_h \rightarrow T^*(D_b)$ is a double cover because each covector in $T^*(D_b)$ is conormal to two ellipses in Y . However, this is not true for \mathcal{R}_h and \mathcal{C}_h and for points in $D_1 \setminus D_b$, and that is why we need the more subtle arguments on D_1 .

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