DEFINITION 4.2.1

Let V be a nonempty set (whose elements are called vectors) on which are defined an addition operation and a scalar multiplication operation with scalars in F. We call V a **vector space over** F, provided the following ten conditions are satisfied:

A1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is also in V. We say that V is **closed under addition**.

A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

A4. *Associativity of addition:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

A6. *Existence of additive inverses in V:* For each vector $\mathbf{v}$ In V, there is a vector, denoted $-\mathbf{v}$, in V such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

A7. *Unit property:* For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

A8. *Associativity of scalar multiplication:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(rs)\mathbf{v} = r(s\mathbf{v}).$$

A9. *Distributive property of scalar multiplication over vector addition:* For all $\mathbf{u}$, $\mathbf{v} \in V$ and all scalars $r \in F$,

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

A10. *Distributive property of scalar multiplication over scalar addition:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}.$$