Inversion of the circular Radon transform on an annulus

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Abstract. The representation of a function by its circular Radon transform (CRT) and various related problems arise in many areas of mathematics, physics and imaging science. There has been a substantial spike of interest towards these problems in the last decade mainly due to the connection between the CRT and mathematical models of several emerging medical imaging modalities. This paper contains some new results about the existence and uniqueness of the representation of a function by its circular Radon transform with partial data. A new inversion formula is presented in the case of the circular acquisition geometry for both interior and exterior problems when the Radon transform is known for only a part of all possible radii. The results are not only interesting as original mathematical discoveries, but can also be useful for applications, e.g. in medical imaging.

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1. Introduction

The circular Radon transform $g = Rf$ puts into correspondence to a given function f its integrals along circles

$$g(x_0, y_0, r) = Rf\ (x_0, y_0, r) = \int_{C(x_0,y_0,r)} f(x, y)\, ds, \tag{1}$$

where $C(x_0, y_0, r)$ denotes the circle of radius r centered at the point (x_0, y_0).

If $Rf(x_0, y_0, r)$ is known for all possible values of its three arguments, then the reconstruction of a function $f(x, y)$ of two variables from Rf is an over-determined problem. It is reasonable to expect that one can still uniquely recover f from Rf after reducing the degrees of freedom of Rf by one. There are many different ways to reduce the dimensions of the data Rf, e.g. by considering only the data coming from circles of a certain fixed radius, circles passing through a fixed point, circles tangent to a line, circles with centers located on a curve, etc. All of these approaches lead to interesting mathematical problems and various research groups have done extensive amount of work on this subject. One can find good surveys and abundant lists of references to papers dedicated to these topics in [3, 4, 17, 18].

In this paper we concentrate on the problem of recovering f from Rf data limited to circles, which are centered on a circle $C(R) \equiv C(0, 0, R)$. Our consideration is partially motivated by several medical imaging applications briefly described below.

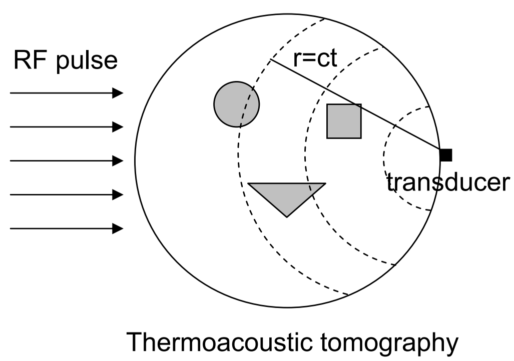

Figure 1. A sketch of TAT/PAT

Thermoacoustic (TAT) and photoacoustic (PAT) tomography are two emerging medical imaging modalities, which are based on the same principles (see [21] for a great survey on mathematical problems in TAT and PAT). The part of the human body being imaged is exposed to a short pulse of electromagnetic (EM) radiation (radio-frequency (RF) waves in TAT, and lasers in PAT). A portion of this radiation is absorbed in the body, heating up the tissue, and causing thermal expansion, which in its turn generates acoustic waves traveling through the body. Multiple transducers placed outside of the body record these acoustic signals during some time. Then the collected data are processed to generate an image of the heat absorption function inside the body. The premise here is that there exists a strong contrast in the amount of absorbed EM energy between different types of tissues. For example cancerous cells absorb several times more energy than the healthy ones, hence recovery of the RF absorption function inside the body can help both to diagnose and to locate cancer. At the same time sound waves have very weak contrast in the tissue, therefore one can simplify the model assuming the sound speed c to be constant in the body. Under this assumption the signals registered by a transducer at any moment of time t would be generated by inclusions lying on a sphere of radius $r = ct$ centered at the transducer location (see Figure 1). Thus the problem of image reconstruction in TAT and PAT boils down to the recovery of the image function f from Rf data along spheres centered at available transducer locations. By using plane-focused transducers one can consider a 2D problem of inverting the circular Radon transform to reconstruct planar slices of the image function. The transducer locations here (i.e. the centers of integration circles) will be limited to a planar curve on the edge of the body. The simplest such curve (i.e. the simplest data acquisition geometry) both for the mathematical model, and from the engineering point of view is a circle, and that is the geometry we consider in this paper (see [18] for a survey on spherical Radon transforms with centers on a sphere).

Another medical imaging modality that uses the circular Radon transform in its mathematical model is the ultrasound reflection tomography (e.g. see [30, 31]). Here the transducer placed at the edge of the body works in dual modes first as an emitter of sound waves, and then as a receiver, registering the reflection of ultrasound waves from the inclusions inside the body. Assuming constant speed of sound propagation, the problem of recovering the reflectivity function inside the body corresponds to the problem of inverting the circular Radon transform with data collected along circles of radius $r = ct/2$. Some other applications of this transform include sonar and radar imaging (e.g. see [9, 26]).

2. Previous works and known results

The major problems studied in relation to Radon transforms include the existence and uniqueness of their inversions, inversion formulae and algorithms, the stability of these algorithms, and the range descriptions of the transforms (e.g. see [14, 28, 29]). The first three problems for the circular transform will be discussed throughout this paper. For the detailed description and known results about range descriptions we refer the reader to papers [2, 5, 19].

The existence and uniqueness problem of the inversion of the circular Radon transform has been studied by many authors for various restrictions of Rf and various classes of function f (see [1, 3, 4, 8, 13, 15, 17, 18, 30, 31] and the references there).

In a classical work [3] Agranovsky and Quinto provided a complete solution to the problem in the case when f has compact support, and Rf is known along circles of all possible radii centered on a given set.

Agranovsky, Berentstein, and Kuchment in [1] used PDE techniques to study the injectivity problem of the spherical Radon transform (n-dimensional generalization of the circular transform) when $f \in L^p(\mathbb{R}^n)$ and Rf is known for spheres of all possible radii centered at every point of the boundary of some domain D.

Finch, Patch, and Rakesh in [17] studied this problem for smooth f supported in a bounded connected domain $D \subset \mathbb{R}^n$. For strictly convex $\overline{D}$ they proved the uniqueness of inversion using Rf from spheres centered on any open subset of the boundary of D and all possible radii. In the case of odd n they also showed uniqueness of inversion from data with spheres centered at every point of the boundary and radii limited to $r < \operatorname{diam}(D)/2$. The proof of the latter result would not extend to even dimensions, since it was based on the solution properties of certain problems related to wave equation, that hold only in odd dimensions.

Ambartsoumian and Kuchment in [4] obtained some further results on injectivity of the spherical transform, providing several sufficient conditions on the data acquisition geometry, in order for the transform to have a unique inverse. That paper also used Rf from spheres of all possible radii.

Lavrentiev, Romanov, and Vasiliev in [25] proved the injectivity of a Radon-type transform integrating along a fairly general family of curves invariant with respect to rotations around the origin, when only half of the possible "radii" of these curves are used, and the function is supported inside the circle. No inversion formulae were derived in that work.

Anastasio et al. showed in [7] that in the 3D spherical acquisition geometry the Radon data for half of all possible radii is sufficient for unique reconstruction of the unknown function supported inside the sphere. It was mentioned that the technique can be applied to obtain a similar result in 2D. The work did not provide an exact inversion formula, and it did not address the uniqueness problem when the support of the unknown function extends outside of the sphere.

In circular acquisition geometry there are various inversion formulae when Rf is known for circles of all possible radii [16, 22, 30]. However, to the best of our knowledge no exact formula is known for the case when Rf is available for only half of all possible radii, or when the support of f is outside the circle.

In this paper we derive inversion formulae of the circular Radon transform from Rf data collected along all circles centered on the circle $C(R)$ and of radii $r < R_1 \leq \operatorname{diam}(D)/2$. The result holds when f is supported inside the annulus $A(\varepsilon, R) = \left\{(x,y) : \varepsilon < \sqrt{x^2+y^2} < R\right\}$ for any $\varepsilon > 0$, as well as when f is supported inside the annulus $A(R, 3R)$. Some other cases, such as f defined inside $D(0, 2R)$ by radial symmetry with respect to the circle $C(R)$, follow as simple corollaries.

3. Main results

3.1. Notations

Throughout this section $f(r,\theta)$ denotes an unknown function supported inside the disc $D(0, 3R)$, where (r,θ) are polar coordinates measured from the center of that disc, and

$R > 0$ is a fixed number. The circular Radon transform of f along a circle of radius ρ centered at a point with polar coordinates (R, ϕ) (see Figure 2) is denoted by

$$g(\rho, \phi) = Rf(\rho, \phi) = \int_{C(\rho,\phi)} f(r, \theta)\, ds. \tag{2}$$

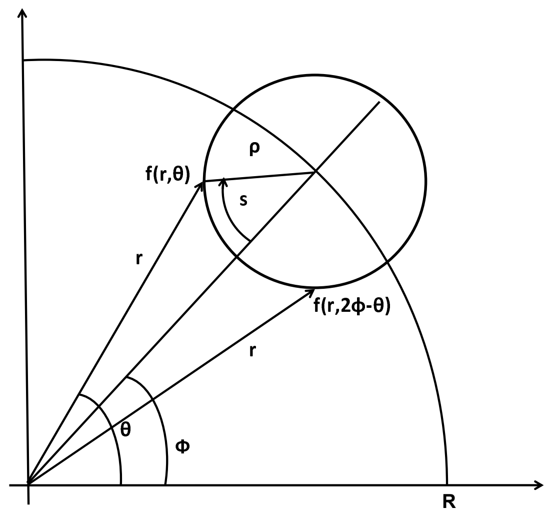

Figure 2. Geometric setup of integration along the circle $C(\rho, \phi)$

The Fourier series generated by $f(r, \theta)$ and $g(\rho, \phi)$ with respect to corresponding angular variables are denoted by

$$f(r, \theta) = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{in\theta}, \tag{3}$$

$$g(\rho, \phi) = \sum_{n=-\infty}^{\infty} g_n(\rho)\, e^{in\phi}, \tag{4}$$

where the Fourier coefficients $f_n(r)$ and $g_n(\rho)$ are computed by

$$f_n(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta)\, e^{-in\theta} d\theta, \tag{5}$$

$$g_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} g(\rho, \phi)\, e^{-in\phi} d\phi. \tag{6}$$

3.2. Functions supported in an annulus $A(\varepsilon, R)$

In this subsection we consider a smooth function $f(r, \theta)$ supported inside the disc of radius R. We will show that the function can be uniquely recovered from Radon data with only part of all possible radii, and then provide a reconstruction formula.

Theorem 1 *Let $f(r, \theta)$ be an unknown continuous function supported inside the annulus $A(\varepsilon, R) = \{(r, \theta) : r \in (\varepsilon, R),\ \theta \in [0, 2\pi]\}$, where $0 < \varepsilon < R$. If $Rf(\rho, \phi)$ is known for $\phi \in [0, 2\pi]$ and $\rho \in [0, R - \varepsilon]$, then $f(r, \theta)$ can be uniquely recovered in $A(\varepsilon, R)$.*

Proof. We will use an approach similar to Cormack's inversion of the linear Radon transform [11]. Let us rewrite formula (2) by considering the contribution dg to $g(\rho,\phi)$ from two equal elements of arc ds of the circle $C(\rho,\phi)$. If the two elements of the arc are located symmetrically with respect to the polar radius of the center of integration circle (see Figure 2), then

$$dg = \sum_{n=-\infty}^{\infty} [f_n(r)\, e^{in\theta} + f_n(r)\, e^{in(2\phi-\theta)}]\, ds, \qquad 0 \le \phi \le \theta \le 2\pi$$

so we can write

$$g(\rho,\phi) = \int_{C^+(\rho,\phi)} \sum_{n=-\infty}^{\infty} [f_n(r)\, e^{in\theta} + f_n(r)\, e^{in(2\phi-\theta)}]\, ds, \qquad 0 \le \phi \le \theta \le 2\pi,$$

where $C^+(\rho,\phi)$ denotes half of the circle $C(\rho,\phi)$ corresponding to $\theta \ge \phi$.

Notice that $e^{in\theta} + e^{in(2\phi-\theta)} = 2\, e^{in\phi} \cos[n(\theta-\phi)]$, and $s = \rho \arccos\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)$, hence

$$ds = \frac{r dr}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}.$$

Exchanging the order of summation and integration and using these relations we get

$$g(\rho,\phi) = \sum_{n=-\infty}^{\infty} 2\, e^{in\phi} \int_{R-\rho}^{R} \frac{f_n(r)\, r \cos[n(\theta-\phi)]}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\, dr$$

Applying $\theta - \phi = \arccos\left(\frac{r^2+R^2-\rho^2}{2rR}\right)$, we obtain

$$g(\rho,\phi) = \sum_{n=-\infty}^{\infty} 2\, e^{in\phi} \int_{R-\rho}^{R} \frac{f_n(r)\, r \cos\left[n \arccos\left(\frac{r^2+R^2-\rho^2}{2rR}\right)\right]}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\, dr \qquad (7)$$

Comparing equations (4) and (7) it is easy to notice that by passing to the basis of complex exponentials we diagonalized the circular Radon transform, i.e. the n-th Fourier coefficient of g depends only on n-th Fourier coefficient of f. This is not surprising, due to rotation invariance property of Rf in the circular geometry. As a result our problem breaks down to the following set of one-dimensional integral equations

$$g_n(\rho) = 2\int_{R-\rho}^{R} \frac{f_n(r)\, r\, T_{|n|}\left(\frac{r^2+R^2-\rho^2}{2rR}\right)}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\, dr, \qquad (8)$$

where $T_k(x)$ is the k-th order Chebyshev polynomial of the first kind.

Let us make a change of variables in the integral (8) by setting $u = R - r$. Then equation (8) becomes

$$g_n(\rho) = \int_0^{\rho} \frac{f_n(R-u)\, 4\rho\, (R-u)\, T_{|n|}\left[\frac{(R-u)^2+R^2-\rho^2}{2R\,(R-u)}\right]}{\sqrt{\rho-u}\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}}\, du. \qquad (9)$$

which can be rewritten as

$$g_n(\rho) = \int_0^\rho \frac{F_n(u)\, K_n(\rho,u)}{\sqrt{\rho-u}}\, du, \tag{10}$$

where

$$F_n(u) = f_n(R-u), \tag{11}$$

$$K_n(\rho,u) = \frac{4\rho\,(R-u)\; T_{|n|}\left[\frac{(R-u)^2+R^2-\rho^2}{2R(R-u)}\right]}{\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}}. \tag{12}$$

Equation (10) is a Volterra integral equation of the first kind with weakly singular kernel (e.g. see [33, 34]). Indeed, due to the assumptions on the support of f we know, that $F_n(u) \equiv 0$ for u close to R or 0. Therefore from formula (12) and the properties of Chebyshev polynomials, it follows that the kernel $K_n(\rho,u)$ is continuous in its arguments (and hence bounded) along with the first order partial derivatives on the support of F_n.

To get rid of the singularity in the kernel of equation(10) we apply the standard method of kernel transformation [35]. Multiplying both sides of equation (10) by $\dfrac{1}{\sqrt{t-\rho}}\, d\rho$ and integrating from 0 to t we get

$$\int_0^t \frac{g_n(\rho)}{\sqrt{t-\rho}}\, d\rho = \int_0^t \int_0^\rho \frac{F_n(u) K_n(\rho,u)}{\sqrt{\rho-u}\sqrt{t-\rho}}\, du\, d\rho, \quad t>0.$$

Changing the order of integration, we obtain

$$\int_0^t \frac{g_n(\rho)}{\sqrt{t-\rho}}\, d\rho = \int_0^t F_n(u)\; Q_n(t,u)\, du, \tag{13}$$

where

$$Q_n(t,u) = \int_u^t \frac{K_n(\rho,u)}{\sqrt{\rho-u}\sqrt{t-\rho}}\, d\rho.$$

The advantage of equation (13) in comparison to equation (10) is that the modified kernel $Q_n(t,u)$ is finite. Indeed, making a change of variables $\rho = u + (t-u)\, l$, $0 \le l \le 1$ in the last integral, we get

$$Q_n(t,u) = \int_0^1 \frac{K_n(u+(t-u)\, l, u)}{\sqrt{l}\,\sqrt{1-l}}\, dl. \tag{14}$$

Since K_n is bounded (say $|K_n| < M$), we obtain

$$|Q_n(t,u)| < M \int_0^1 \frac{dl}{\sqrt{l}\,\sqrt{1-l}} = M\pi.$$

In addition $Q_n(t,t) = \pi K_n(t,t) = \pi \sqrt{\dfrac{2t\,(R-t)}{R}} \neq 0$ on the support of F_n.

Now we can easily modify equation (13) to a Volterra equation of second kind. Differentiating equation (13) with respect to t we get

$$\frac{d}{dt}\int_0^t \frac{g_n(\rho)}{\sqrt{t-\rho}}\, d\rho = \pi F_n(t)\; K_n(t,t) + \int_0^t F_n(u)\; \left[\frac{\partial}{\partial t}\int_u^t \frac{K_n(\rho,u)}{\sqrt{\rho-u}\sqrt{t-\rho}}\, d\rho\right] du.$$

Dividing both sides of the last equation by $\pi K_n(t,t)$ and denoting

$$G_n(t) = \frac{1}{\pi K_n(t,t)} \frac{d}{dt} \int_0^t \frac{g_n(\rho)}{\sqrt{t-\rho}}\, d\rho, \tag{15}$$

and

$$L_n(t,u) = \frac{1}{\pi K_n(t,t)} \frac{\partial}{\partial t} \int_u^t \frac{K_n(\rho,u)}{\sqrt{\rho-u}\sqrt{t-\rho}}\, d\rho \tag{16}$$

we finally obtain a Volterra equation of second kind

$$G_n(t) = F_n(t) + \int_0^t F_n(u)\, L_n(t,u)\, du, \tag{17}$$

where the kernel $L_n(t,u)$ is continuous on the support of F_n. To see the continuity of L_n one can make a change of variables in equation (16)

$$\rho = t\cos^2\beta + u\sin^2\beta, \quad \beta \in [0,\pi/2],$$

and express L_n as

$$L_n(t,u) = \frac{2}{\pi\, K_n(t,t)} \frac{\partial}{\partial t} \int_0^{\pi/2} K_n(t\cos^2\beta + u\sin^2\beta, u)\, d\beta.$$

The Volterra equation of the second kind (17) has a unique solution, which finishes the proof of the theorem. □

Using the Picard's process of successive approximations (e.g. see [34]) for the solution of Volterra equations of second kind one can immediately obtain the following

Corollary 2 *An exact solution of equation (17) is given by the formula*

$$F_n(t) = G_n(t) + \int_0^t H_n(t,u)\, G_n(u)\, du, \tag{18}$$

where the resolvent kernel $H_n(t,u)$ is given by the series of iterated kernels

$$H_n(t,u) = \sum_{i=1}^{\infty} (-1)^i L_{n,i}(t,u), \tag{19}$$

defined by

$$L_{n,1}(t,u) = L_n(t,u), \tag{20}$$

and

$$L_{n,i}(t,u) = \int_u^t L_{n,1}(t,x)\, L_{n,i-1}(x,u)\, dx, \quad \forall\, i \geq 2. \tag{21}$$

This corollary (with notations defined in formulas (11), (12), (15), (16)) provides a new exact formula for inversion of the circular Radon transform in circular acquisition geometry. Its advantage compared to all the other known exact inversion formulas is the fact that only part of the Rf data is used. Namely, it is easy to notice the following

Remark 3 *In order to reconstruct the function $f(r,\theta)$ in any subset Ω of the disc of its support $D(0,R)$, the inversion formula in Corollary 2 requires the knowledge of $Rf(\rho,\phi)$ only for $\rho < R - R_0$, where $R_0 = \inf\{|x|,\ x \in \Omega\}$.*

In medical imaging reducing the radial data redundancy can be essential for increasing the depth and reducing the time of imaging.

Remark 4 *The resolvent kernel $H_n(t,u)$ is the same for any functions f and g. Hence in practice one needs to compute it with the desired accuracy only once, and then it can be used with any data set.*

Remark 5 *In Theorem 1 we require f to be continuous, which guarantees the convergence of the Fourier series (3) and (4) almost everywhere. If one needs to ensure convergence everywhere, then some additional conditions on f (e.g. bounded variation) should be added in Theorem 1 and the other two theorems in this paper.*

3.3. Functions supported inside an annulus $A(R, 3R)$

Let us now consider an exterior problem in the circular acquisition geometry, i.e. the Radon data is still collected along circles centered on a circle of radius R, however the unknown function f is supported outside of the disc $D(0,R)$.

Theorem 6 *Let $f(r,\theta)$ be an unknown continuous function supported inside the annulus $A(R,\,3R) = \{(r,\theta) : r \in (R,\,3R),\ \theta \in [0,2\pi]\}$. If $Rf(\rho,\phi)$ is known for $\phi \in [0,2\pi]$ and $\rho \in [0,R_1]$, where $0 < R_1 < 2R$ then $f(r,\theta)$ can be uniquely recovered in $A(R,\,R_1)$.*

Proof.
The argument of the proof of the previous theorem repeats here with very small changes. The condition $0 < R_1 < 2R$ guarantees that all integration circles $C(\rho,\phi)$ intersect the boundary of the disc $D(0,R)$. Hence equation (7) in this case becomes

$$g(\rho,\phi) = \sum_{n=-\infty}^{\infty} 2\, e^{in\phi} \int_R^{R+\rho} \frac{f_n(r)\, r \cos\left[n \arccos\left(\frac{r^2+R^2-\rho^2}{2rR}\right)\right]}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\, dr \tag{22}$$

Then in a similar way, we have

$$g_n(\rho) = 2\int_R^{R+\rho} \frac{f_n(r)\; r\; T_{|n|}\left(\frac{r^2+R^2-\rho^2}{2rR}\right)}{R\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\, dr, \tag{23}$$

Now making a change of variables $u = r - R$ in the last expression we get

$$g_n(\rho) = \int_0^{\rho} \frac{f_n(R+u)\, 4\rho\, (R+u)\; T_{|n|}\left[\frac{(R+u)^2+R^2-\rho^2}{2R\,(R+u)}\right]}{\sqrt{\rho-u}\sqrt{(u+\rho)(2R+u+\rho)(2R+u-\rho)}}\, du. \tag{24}$$

which can be rewritten as

$$g_n(\rho) = \int_0^{\rho} \frac{F_n(u)\, K_n(\rho,u)}{\sqrt{\rho-u}}\, du, \tag{25}$$

where

$$F_n(u) = f_n(R+u), \tag{26}$$

$$K_n(\rho, u) = \frac{4\rho\,(R+u)\,T_{|n|}\left[\frac{(R+u)^2+R^2-\rho^2}{2R\,(R+u)}\right]}{\sqrt{(u+\rho)(2R+u+\rho)(2R+u-\rho)}}. \tag{27}$$

Notice, that if one would allow $\rho > 2R$, then $K_n(\rho, u)$ would become unbounded due to the last multiplier in the denominator. This shows that $3R$ is an accurate upper limit for the outer radius of the annulus in the hypothesis of the theorem.

In analogy with the proof of the previous theorem we get

$$K_n(t,t) = \sqrt{\frac{2t\,(R+t)}{R}} \neq 0.$$

All the other steps literally repeat the proof of Theorem 1. □

3.4. Functions supported inside the disc $D(0, 2R)$

It is easy to note that in some special cases one can combine the results of the previous two theorems to reconstruct a function whose support is located both inside and outside of the circular path $C(R)$ of data acquisition. For example

Theorem 7 *Let f be an unknown continuous function supported inside the disc $D(0, 2R)$. Assume also that $f \equiv 0$ in some neighborhood of the circle $C(R)$, and all its Fourier coefficients are even (or odd) with respect to $C(R)$, i.e. $f_n(R+u) = f_n(R-u)$ (or $f_n(R+u) = -f_n(R-u)$) for $\forall u \in [0, R]$. If $Rf(\rho, \phi)$ is known for $\phi \in [0, 2\pi]$ and $\rho \in [0, R_1]$, where $0 < R_1 < R$ then $f(r, \theta)$ can be uniquely recovered in $A(R - R_1,\, R + R_1)$.*

Proof. Combining the two previous results, we obtain a Volterra integral equation of the first kind

$$g_n(\rho) = \int_0^\rho \frac{F_n(u)K_n(\rho, u)}{\sqrt{\rho - u}}\,du \tag{28}$$

where

$$F_n(u) = f_n(R+u), \tag{29}$$

and

$$K_n(\rho, u) = \tag{30}$$

$$\frac{4\rho}{\sqrt{u+\rho}} \left\{ \frac{(R+u)\,T_{|n|}\left[\frac{(R+u)^2+R^2-\rho^2}{2R\,(R+u)}\right]}{\sqrt{(2R+u+\rho)(2R+u-\rho)}} \pm \frac{(R-u)\,T_{|n|}\left[\frac{(R-u)^2+R^2-\rho^2}{2R\,(R-u)}\right]}{\sqrt{(2R+\rho-u)(2R-\rho-u)}} \right\}$$

The rest of the proof is carried out a long the same lines as before. □

It is interesting to note that the circular Radon transform in the linear acquisition geometry can be uniquely inverted on the class of continuous functions that are even with respect to the linear path of the data acquisition. At the same time all odd functions are mapped to zero by that transform. In our case (of circular acquisition geometry), the circular Radon transform can be uniquely inverted on classes of functions with Fourier coefficients that are even with respect to the circular path of data acquisition, as well as with the ones that are odd.

4. Additional remarks

(i) While it is well-established that acoustic breast tomography in its various forms is a classic example of spherical Radon-based imaging *inside* a spherical/circular (3D/2D) aperture, the case of imaging *outside* a spherical aperture is a less well-described biomedical concept. Two biomedical imaging methods can currently be modeled in the time domain through spherical transforms of a function exterior to the aperture: transrectal ultrasound (TRUS) [32] and intravascular ultrasound (IVUS) (see Section 8.10 in [10] and the references there). In both TRUS and IVUS, a ultrasound array arranged on the surface of a cylinder is introduced into the body with the goal of producing a transverse or axial image. In TRUS, the typical application is imaging of the male prostate, while IVUS is a higher resolution ultrasound technique typically used to evaluate plaques in blood vessels.
A setup where the support of the unknown function is located on both sides of the data acquisition path may not be relevant to medical imaging, however it can be applicable in radar and sonar imaging [9, 26].

(ii) The reconstruction of f from partial Rf data is an extremely ill-conditioned problem and, despite all uniqueness results, in practical implementation one can expect to recover stably only certain parts of the image, the rest of it being blurred out (e.g., [6, 39]). This is due to the fact that some parts of the wavefront set $WF(f)$ of the image will be lost ([26, 40]). More specifically, a point $(x, \xi) \in WF(f)$ can be stably detected from the Radon data, if and only if Rf includes data obtained from a circle passing through x and co-normal to ξ. In other words, one can see only those parts of image singularities that can be tangentially touched by available circles of integration.
It is easy to notice that in the case of the interior problem with available radii $\rho < R$ all singularities can be stably resolved. In the exterior problem only singularities in the directions close to normals of polar radius can be recovered with little or no blurring. However, this may be enough for example in IVUS, where the imaging is done along the vein walls, which are normal to the polar radii directions.

(iii) The main results of the paper have potential to be generalized to higher dimensions using spherical harmonics and Gegenbauer polynomials akin to the generalization of Cormack's original inversion [11] to higher dimensions (e.g. see [27] and [12]). The authors plan to address this issue in future work.

(iv) An accurate and efficient numerical implementation of the inversion formulae derived in the paper is an interesting problem in its own right. This includes, among other things, a careful study of conditions ensuring that the resolvent kernels (19) are bounded. The authors plan to address this problem in future work.

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