

A RANGE DESCRIPTION FOR THE PLANAR CIRCULAR RADON TRANSFORM*

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Abstract. The transform considered in the paper integrates a function supported in the unit disk on the plane over all circles centered at the boundary of this disk. Such a circular Radon transform arises in several contemporary imaging techniques, as well as in other applications. As is common for transforms of Radon type, its range has infinite codimension in standard function spaces. Range descriptions for such transforms are known to be very important for computed tomography—for instance, when dealing with incomplete data, error correction, and other issues. A complete range description for the circular Radon transform is obtained. Range conditions include the recently found set of moment-type conditions, which happens to be incomplete, as well as other conditions that have less standard form. In order to explain the procedure better, a similar (nonstandard) treatment of the range conditions is described first for the usual Radon transform on the plane.

Key words. thermoacoustic tomography, Radon transform, range description

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1. Introduction. The following “circular” Radon transform, which is the main object of study in this article, arises in several applications, including the newly developing thermoacoustic tomography and its sibling, optoacoustic tomography (e.g., [5, 16, 29, 56, 67, 68, 69, 70]), as well as radar, sonar, and other applications [40, 48, 51, 52]. It has also been considered in relation to some problems of approximation theory, mathematical physics, and other areas [1, 2, 10, 13, 28, 31, 38, 39].

Let $f(x)$ be a continuous function on \mathbb{R}^d , $d \geq 2$.

DEFINITION 1. *The circular Radon transform of f is defined as*

$$Rf(p, \rho) = \int_{|y-p|=\rho} f(y) d\sigma(y),$$

where $d\sigma(y)$ is the surface area on the sphere $|y - p| = \rho$ centered at $p \in \mathbb{R}^d$.

In this definition we do not restrict the set of centers p or radii r . It is clear, however, that this mapping is overdetermined, since the dimension of pairs (p, r) is $d+1$, while the function f depends on d variables only. This (as well as the tomographic motivation) suggests that we restrict the set of centers to a set (hypersurface) $S \subset \mathbb{R}^d$, while not imposing any restrictions on the radii. This restricted transform is denoted by R_S :

$$R_S f(p, \rho) = Rf(p, \rho)|_{p \in S}.$$

In this paper we will be dealing with the planar case only, i.e., the dimension d will be equal to 2. Due to tomographic applications, where S is the set of locations of

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transducers [29, 67, 68, 70], from now on we will be looking at the specific case when S is the unit circle $|x| = 1$ in the plane.

There are many questions one can ask concerning the circular transform R_S , e.g., concerning its injectivity, inversion formulas, stability of inversion, and range description. Experience in computerized tomography shows (e.g., [44, 46]) that all these questions are important. Although none of them has been resolved completely for R_S , significant developments have occurred recently (see, e.g., [1, 2, 5, 6, 12, 14, 16, 20, 35, 46, 47, 51, 52, 55, 59, 60, 67, 68, 69]). The goal of this article is to describe the range of R_S in the two-dimensional case, with S being the unit circle. Moreover, we will be dealing with functions supported inside the circle S only. The properties of the operator R_S (e.g., stability of the inversion, its Fourier integral operator (FIO) properties, etc.) deteriorate on functions with supports extending outside S (e.g., [2, 16, 70] and remarks in the last section of this article). However, in tomographic applications one normally deals with functions supported inside S only [29, 56, 67, 70].

As was already mentioned, the range of R_S has infinite codimension (e.g., in spaces of smooth functions, see details below) and thus infinitely many range conditions appear. It seems to be a rather standard situation for various types of Radon transforms that range conditions split into two types, one of which is usually easier to discover, while the other “half” is harder to come by. For instance, it took about a decade to find the complete range description for the so-called exponential Radon transform arising in SPECT (single photon emission computed tomography) [3, 4, 33, 34, 66]. For a more general attenuated transform arising in SPECT, it took twice as much time to move from a partial set of range conditions [44, 45] to the complete set [53]. In the circular case, a partial set of such conditions was discovered recently [56]. It happens to be incomplete, and the goal of this text is to find the complete one.

One might ask why is it important to know the range conditions. Such conditions have been used extensively in tomography (as well as in radiation therapy planning, e.g., [8, 9, 30]) for various purposes: completing incomplete data, detecting and correcting measurement errors and hardware imperfections, recovering unknown attenuation, etc. [26, 41, 42, 43, 44, 49, 50, 57, 64, 65]. Thus, as soon as a new Radon-type transform arises in an application, a quest for the range description begins.

In order to explain our approach, we start in the next section with treating a toy example of the standard Radon transform on the plane, where the range conditions are well known (e.g., [13, 17, 18, 19, 25, 44, 46]). Our approach, however, is different from the standard ones and naturally leads to the considerations of the circular transform in the rest of the paper.

2. The case of the planar Radon transform. In this section we will approach in a somewhat nonstandard way the issue of the range description for the standard Radon transform on the plane. Consider a compactly supported smooth function $f(x)$ on the plane and its Radon transform

$$(1) \quad (\mathcal{R}f)(\omega, s) = g(\omega, s) := \int_{x \cdot \omega = s} f(x) dl,$$

where $s \in \mathbb{R}$, $\omega \in S^1$ is a unit vector in \mathbb{R}^2 , and dl is the arc length measure on the line $x \cdot \omega = s$. We want to describe the range of this transform, say on the space $C_0^\infty(\mathbb{R}^2)$. Such a description is well known (see, e.g., [13, 17, 18, 19, 25, 44, 46], or any other book or survey on Radon transforms or computed tomography).

THEOREM 2. *A function g belongs to the range of the Radon transform on C_0^∞ if and only if the following conditions are satisfied:*

1. $g \in C_0^\infty(S^1 \times \mathbb{R})$;
2. for any $k \in \mathbb{Z}^+$ the k th moment $G_k(\omega) = \int_{-\infty}^\infty s^k g(\omega, s) ds$ is the restriction to the unit circle S^1 of a homogeneous polynomial of ω of degree k ;
3. $g(\omega, s) = g(-\omega, -s)$.

We would like to look at this result from a slightly different perspective, which will allow us to do a similar thing in the case of the circular Radon transform.

In order to do so, let us expand $g(\omega, s)$ into the Fourier series with respect to the polar angle ψ (i.e., $\omega = (\cos \psi, \sin \psi)$):

$$(2) \quad g(\omega, s) = \sum_{n=-\infty}^\infty g_n(s) e^{in\psi}.$$

We can now reformulate the last theorem in the following, somewhat unusual way.

THEOREM 3. *A function g belongs to the range of the Radon transform on C_0^∞ if and only if the following conditions are satisfied:*

1. $g \in C_0^\infty(S^1 \times \mathbb{R})$;
2. for any n , the Mellin transform $Mg_n(\sigma) = \int_0^\infty s^{\sigma-1} g_n(s) ds$ of the n th Fourier coefficient g_n of g vanishes at any pole σ of the function $\Gamma(\frac{\sigma+1-|n|}{2})$;
3. $g(\omega, s) = g(-\omega, -s)$.

Since the only difference in the statements of these two theorems is in conditions 2, let us check that these conditions mean the same thing in both cases. Indeed, let us expand $g(\omega, s)$ into Fourier series (2) with respect to ψ . Representing $e^{in\psi}$ as the homogeneous polynomial $(\omega_1 + i(\text{sign } n)\omega_2)^{|n|}$ of ω of degree $|n|$, and noticing that $\omega_1^2 + \omega_2^2 = 1$ on the unit circle, one easily concludes that condition 2 in Theorem 2 is equivalent to the following: the k th moment $\int_{\mathbb{R}} s^k g_n(s) ds$ of the n th Fourier coefficient vanishes for integers $0 \leq k < |n|$ such that $k - n$ is even.

Let us now look at condition 2 in Theorem 3, still using the same Fourier expansion. Notice that when $k - |n|$ is a negative even integer, $Mg_n(\sigma)$ is one-half of the moment of order $k = \sigma - 1$ of g_n . Taking into account that $\Gamma(\frac{\sigma+1-|n|}{2}) = \Gamma(\frac{k+2-|n|}{2})$ has poles exactly when $k - |n|$ is a negative even integer, we see that conditions 2 in both theorems are in fact saying the same thing.

One can now ask the question, why should one disguise in the statement of Theorem 3 negative integers as poles of Gamma-function and usual moments as values of Mellin transforms? The answer is that in the less invariant and thus more complex situation of the circular Radon transform, one can formulate a range description in the spirit of Theorem 3, although it is unclear how to get an analogue of the version given in Theorem 2.

As a warm-up, let us derive condition 2 in Theorem 3 directly, without relying on the version given in the preceding theorem. This is in fact an easy by-product of Cormack's inversion procedure; see, e.g., [46, section II.2]. Indeed, if we write down the original function $f(x)$ in polar coordinates $r(\cos \phi, \sin \phi)$ and expand into the Fourier series with respect to the polar angle ϕ ,

$$(3) \quad f(r(\cos \phi, \sin \phi)) = \sum_{n=-\infty}^\infty f_n(r) e^{in\phi},$$

then the Fourier coefficients f_n and g_n of the original and of its Radon transform are

related as follows [44, formula (2.17) and further]:

$$(4) \quad M(r f_n(r))(s) = \frac{(M g_n)(s)}{B_n(s)},$$

where

$$(5) \quad B_n(s) = \text{const} \frac{\Gamma(s)2^{-s}}{\Gamma((s+1+|n|)/2)\Gamma((s+1-|n|)/2)}.$$

Thus, condition 2 of Theorem 3 guarantees that the function $M(r f_n(r))(s)$ does not develop singularities (which it cannot do for a C_0^∞ -function f) at zeros of $B_n(s)$. It is not that hard now to prove also sufficiency in the theorem, applying Cormack’s inversion procedure to g satisfying conditions 1–3. However, we are not going to do so, since in the next sections we will devote ourselves to doing a similar thing in the more complicated situation of the circular Radon transform.

3. The circular Radon transform. Formulation of the main result. Let us recall the notion of the Hankel transform (e.g., [11]). For a function $h(r)$ on \mathbb{R}^+ , one defines its Hankel transform of an integer order n as follows:

$$(6) \quad (\mathcal{H}_n h)(\sigma) = \int_0^\infty J_n(\sigma r)h(r)r \, dr,$$

where the standard notation J_n is used for Bessel functions of the first kind.

As in the introduction, let R_S be the circular Radon transform on the plane that integrates functions compactly supported inside the unit disk D over all circles $|x - p| = \rho$ with centers p located on the unit circle $S = \{p \mid |p| = 1\}$. Since this transform commutes with rotations about the origin, the Fourier series expansion with respect to the polar angle partially diagonalizes the operator, and thus the n th Fourier coefficient $g_n(\rho)$ of $g = R_S f$ will depend on the n th coefficient f_n of the original f only. It was shown in [51] that the following relation between these coefficients holds:

$$(7) \quad g_n(\rho) = 2\pi\rho\mathcal{H}_0\{J_n\mathcal{H}_n\{f_n\}\}.$$

For the reader’s convenience, we will provide the brief derivation from [51]. Considering a single harmonic $f = f_n(r)e^{in\phi}$ and using polar coordinates, one obtains

$$(8) \quad g_n(\rho) = \int_0^\infty r f_n(r)dr \int_0^{2\pi} \delta\left[(r^2 + 1 - 2r \cos \phi)^{1/2} - \rho\right] e^{-in\phi} d\phi.$$

Thus, the computation boils down to evaluating the integral

$$I = \int_0^{2\pi} \delta\left[(r^2 + 1 - 2r \cos \phi)^{1/2} - \rho\right] e^{-in\phi} d\phi.$$

Using the standard identity

$$\delta(\rho' - \rho) = \rho \int_0^\infty J_0(\rho'z)J_0(\rho z)z \, dz$$

and the identity that is easy to obtain from one of the addition formulas, e.g., from [7, formula (4.10.6)]

$$2\pi J_n(az)J_n(bz) = \int_0^{2\pi} J_0[z(a^2 + b^2 - 2ab \cos \phi)^{1/2}]e^{-in\phi} d\phi,$$

one goes from (8) to (7).

Since Hankel transforms are involutive, it is easy to invert (7) and get Norton’s inversion formulas [51]

$$(9) \quad f_n = \frac{1}{2\pi} \mathcal{H}_n \left\{ \frac{\mathcal{H}_0\{g_n(\rho)/\rho\}}{J_n} \right\}.$$

Now one can clearly see analogies to the case of the Radon transform, where zeros of Bessel functions should probably introduce some range conditions. This happens to be correct and leads to the main result of this article, as follows.

THEOREM 4. *In order for the function $g(p, \rho)$ on $S^1 \times \mathbb{R}$ to be representable as $R_S f$ with $f \in C_0^\infty(D)$, it is necessary and sufficient that the following conditions be satisfied:*

1. $g \in C_0^\infty(S^1 \times (0, 2))$.
2. For any n , the $2k$ th moment $\int_0^\infty \rho^{2k} g_n(\rho) d\rho$ of the n th Fourier coefficient of g vanishes for integers $0 \leq k < |n|$. (Equivalently, the $2k$ th moment $\int_0^\infty \rho^{2k} g(p, \rho) d\rho$ is the restriction to the unit circle S of a (nonhomogeneous) polynomial of p of degree at most k .)
3. For any $n \in \mathbb{Z}$, function $\mathcal{H}_0\{g_n(\rho)/\rho\}(\sigma) = \int_0^\infty J_0(\sigma\rho)g_n(\rho)d\rho$ vanishes at any zero $\sigma \neq 0$ of Bessel function J_n . (Equivalently, the n th Fourier coefficient with respect to $p \in S^1$ of the “Bessel moment” $G_\sigma(p) = \int_0^\infty J_0(\sigma\rho)g(p, \rho)d\rho$ vanishes if $\sigma \neq 0$ is a zero of Bessel function J_n .)

4. Proof of the main result. Let us start by proving necessity, which is rather straightforward. Indeed, the necessity of condition 1 is obvious. Let us prove the second condition. In fact, it has already been established in [56]. Let us repeat for completeness its simple proof. Let k be an integer. Consider the moment of order $2k$ of g :

$$(10) \quad \int_0^\infty \rho^{2k} g(p, \rho) d\rho = \int_{\mathbb{R}^2} |x - p|^{2k} f(x) dx = \int_{\mathbb{R}^2} (|x|^2 - 2x \cdot p + 1)^k f(x) dx$$

(we have taken into account that $|p| = 1$). We see that the resulting expression is the restriction to S^1 of a (nonhomogeneous) polynomial of degree k in variable p . Expanding into Fourier series with respect to the polar angle of p , we see that the n th harmonic g_n contributes the following homogeneous polynomial of degree $|n|$ in the variable p :

$$\left(\int_0^\infty \rho^{2k} g_n(\rho) d\rho \right) e^{in\psi}.$$

Here as before $p = (\cos \psi, \sin \psi)$. Thus, for $|n| > k$, this term must vanish, which gives necessity of condition 2. We will return to a discussion of this condition below to add a new twist to it.

Necessity of condition 3 follows immediately from Norton’s formula (9), which implies in particular that

$$\mathcal{H}_0\{g_n(\rho)/\rho\} = 2\pi J_n \mathcal{H}_n\{f_n\}.$$

Since both functions J_n and $\mathcal{H}_n\{f_n\}$ are entire, $\mathcal{H}_0\{g_n(\rho)/\rho\}$ vanishes whenever J_n does.

Remark 5. The reader might ask why in the third condition of the theorem we do not take into account the zero root of J_n , which in fact has order n , while nonzero

roots are all simple. The reason is that condition 2 already guarantees that $\sigma = 0$ is zero of order $2n$ of $\mathcal{H}_0\{g_n(\rho)/\rho\}$ (twice higher than that of J_n). Indeed, due to evenness of J_0 , function $\mathcal{H}_0\{g_n(\rho)/\rho\}(\sigma)$ is also even. Thus, all odd order derivatives at $\sigma = 0$ vanish. The known Taylor expansion of J_0 at zero leads to the formula

$$\mathcal{H}_0\{g_n(\rho)/\rho\}(\sigma) = \sum_m \frac{(-1)^m}{(m!)^2} \left(\frac{\sigma}{2}\right)^{2m} \int_0^\infty r^{2m} g_n(r) dr.$$

We see now that the moment condition 2 guarantees that $\sigma = 0$ is zero of order $2n$ of $\mathcal{H}_0\{g_n/\rho\}(\sigma)$.

Let us move to the harder part, proving sufficiency. Assume a function g satisfies conditions of the theorem and is supported in $S \times (\epsilon, 2 - \epsilon)$ for some positive ϵ . We will show that then $g = R_S f$ for some $f \in C_0^\infty(D_\epsilon)$, where D_ϵ is the disk $|x| < 1 - \epsilon$ in the plane.

Due to Norton’s formulas, it is natural to expect the proof to go along the following lines: expand g into the Fourier series $g = \sum_m g_m(\rho) e^{im\psi}$ with respect to the angle variable ψ , then use (9) to construct a function f and then show that f is of an appropriate function class and that its circular Radon transform is equal to g . This is what we are going to do, with a small caveat: instead of constructing f itself, we will construct its two-dimensional Fourier transform. Besides, we will start considering the partial sums of the series $h_n = \sum_{|m| \leq n} g_m(\rho) e^{im\phi}$. But first we need to get some simple estimates from below for the Bessel function of the first kind J_n .

LEMMA 6. *On the entire complex plane, except for a disk S_0 centered at the origin and a countable number of disks S_k of radii $\pi/6$ centered at points $\pi(k + \frac{2n+3}{4})$, one has*

$$(11) \quad |J_n(z)| \geq \frac{C e^{|\text{Im } z|}}{\sqrt{|z|}}, \quad C > 0.$$

Proof. Let us split the complex plane into three parts by a circle S_0 of a radius R (to be chosen later) centered at the origin and a planar strip $\{z = x + iy \mid |y| < a\}$, as follows: part I consists of points z satisfying $|z| \geq R$ and $|\text{Im } z| \geq a$; part II consists of points such that $|z| \geq R$ and $|\text{Im } z| < a$; part III is the interior of S_0 , i.e., $|z| < R$. It is clearly sufficient to prove the estimate (11) in the first two parts: outside and inside the strip. Using the parity property of J_n , it suffices to consider only the right half plane $\text{Re } z \geq 0$.

The Bessel function of the first kind $J_n(z)$ has the following known asymptotic representation in the sector $|\arg z| \leq \pi - \delta$ (e.g., [7, formula (4.8.5)] or [36, formula (5.11.6)]):

$$(12) \quad \begin{aligned} J_n(z) = & \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) (1 + O(|z|^{-2})) \\ & - \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) \left(\frac{4n^2 - 1}{8z} + O(|z|^{-3})\right). \end{aligned}$$

Let us start estimating in the first part of the complex plane, i.e., where $|\text{Im } z| > a$ and $|z| > R$ for sufficiently large a and R (and, as we have agreed, $\text{Re } z \geq 0$). There, due to boundedness of $\tan z$ in this region, one concludes that $\frac{\sin z}{z} = \cos z (O(|z|^{-1}))$, and thus (12) implies

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) (1 + O(|z|^{-1})),$$

which in turn for sufficiently large a, R leads to

$$(13) \quad |J_n(z)| \geq \frac{C e^{|\operatorname{Im} z|}}{\sqrt{|z|}}.$$

In the second part of the plane (right half of the strip), due to boundedness of $\sin z$ we have

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \left[\cos \left(z - \frac{\pi n}{2} - \frac{\pi}{4} \right) (1 + O(|z|^{-2})) + O(|z|^{-1}) \right].$$

Consider the system of nonintersecting circles S_k with centers at $z_k = \frac{\pi}{2} + k\pi + \frac{\pi n}{2} + \frac{\pi}{4}$ and radii equal to $\frac{\pi}{6}$. Then outside these circles $|\cos(z - \frac{\pi n}{2} - \frac{\pi}{4})| \geq C$ and

$$|J_n(z)| \geq \frac{C}{\sqrt{|z|}} (1 + O(|z|^{-1})).$$

This implies that for a suitably chosen and sufficiently large R , inside of the strip and outside the circles S_k , we have

$$(14) \quad |J_n(z)| \geq \frac{C e^{|\operatorname{Im} z|}}{\sqrt{|z|}}$$

for $|z| > R$. This proves the statement of the lemma. \square

Let us now return to our task: consider the function g and the partial sums h_n of its Fourier series.

LEMMA 7.

1. If $g(\phi, \rho) = \sum_m g_m(\rho) e^{im\psi}$ satisfies conditions of Theorem 4 and is supported in $S \times (\epsilon, 2 - \epsilon)$, then each partial sum $h_n = \sum_{|m| < n} g_m(\rho) e^{im\psi}$ does so.
2. For any n , h_n is representable as $R_S f_n$ for a function $f_n \in C_0^\infty(D_\epsilon)$.

Proof. The first statement of the lemma is obvious.

Thus, it is sufficient to prove the second statement for a single term $g = g_n(\rho) e^{in\psi}$. As was just mentioned, we will reconstruct the Fourier transform F of the function f . In order to do this, we will use the standard relation between Fourier and Hankel transforms. As before, let $f(x) = f_n(r) e^{in\phi}$, where $r = |x|$ and ϕ are polar coordinates on \mathbb{R}^2 . Then the Fourier transform $F(\xi)$ of f at points of the form $\xi = \sigma\omega$, where $\sigma \in \mathbb{C}$ and $\omega = (\cos \psi, \sin \psi) \in \mathbb{R}^2$ can be written up to a constant factor as follows:

$$(15) \quad F(\sigma\omega) = \mathcal{H}_n(f_n)(\sigma) e^{in\psi}$$

(e.g., [11, end of section 14.1]). If we knew that $g = R_S f$, then according to (7) this would mean that

$$(16) \quad F(\sigma\omega) = F(\sigma) e^{in\psi} = \frac{1}{2\pi} \frac{\mathcal{H}_0(g_n(\rho)/\rho)(\sigma)}{J_n(\sigma)} e^{in\psi}.$$

Let us now take (16) as the definition of $F(\sigma\omega)$. Due to the standard parity property of Bessel functions, such an F is a correctly defined function of $\sigma\omega$ for $\sigma \neq 0$ (i.e., $F(\sigma\omega) = F((-\sigma)(-\omega))$). We would like to show that it is the Fourier transform of a function $f \in C_0^\infty(D_\epsilon)$. Let us prove first that F belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. In order to do so, we need to show its smoothness with respect to the angular variable ψ ; smoothness and fast decay with all derivatives in the radial variable σ ; as

well as that no singularity arises at the origin, which in principle could, due to usage of polar coordinates. Smoothness with respect to the angular variable is obvious, due to (16). Let us deal with the more complex issue of smoothness and decay with respect to σ . First of all, taking into account that $g_n(\rho)$ is supported inside $(0, 2)$, and due to the standard two-dimensional Paley–Wiener theorem, we conclude that $u(\sigma) = \mathcal{H}_0(g_n(\rho)/\rho)$ is an entire function that satisfies for any N the estimate

$$(17) \quad |u(\sigma)| \leq C_N(1 + |\sigma|)^{-N} e^{(2-\epsilon)|\operatorname{Im} \sigma|}.$$

According to the range conditions 2 and 3 of the theorem, this function vanishes at all zeros of Bessel function $J_n(\sigma)$ at least to the order of the corresponding zero. This means that function $F(\sigma) = \frac{u(\sigma)}{2\pi J_n(\sigma)}$ is entire. Let us show that it belongs to a Paley–Wiener class.

Indeed, $\mathcal{H}(g_n(\rho)/\rho)$ is an entire function with Paley–Wiener estimate (17). Due to the estimate from below for J_n (11) given in Lemma 6, we conclude that $F(\sigma\omega)$ is an entire function of Paley–Wiener class in the radial directions, uniformly with respect to the polar angle. Namely,

$$(18) \quad |F(\sigma)| \leq C_N(1 + |\sigma|)^{-N} e^{(1-\epsilon)|\operatorname{Im} \sigma|}.$$

Indeed, outside the family of circles S_k the estimate (11) together with (17) gives the Paley–Wiener estimate (18) we need. Inside these circles, application of the maximum principle finishes the job. Smoothness with respect to the polar angle is obvious. Thus, the only thing one needs to establish to verify that F belongs to the Schwartz class is that F is smooth at the origin. This, however, is the standard question in the Radon transform theory, the answer to which is well known (e.g., [17, pp. 108–109], [18, 19], [25, Chap. 1, proof of Theorem 2.4]). Namely, one needs to establish that for any nonnegative integer k , the k th radial (i.e., with respect to σ) derivative of $F(\sigma\omega)$ at the origin is a homogeneous polynomial of order k with respect to ω . So, let us check that this condition is satisfied in our situation. First of all, the parity of the function F is the same as that of n . Thus, we do not need to worry about the derivatives $F_\sigma^{(k)}|_{\sigma=0}$ with $k - n$ odd, since they are zero automatically. Due to the special single-harmonic form of F , we only need to check that $F_\sigma^{(k)}|_{\sigma=0} = 0$ for $k < |n|$ with $k - n$ even. This, however, as we have already discussed in Remark 5, follows from the moment conditions 2 of the theorem.

Due to the smoothness that we have just established and Paley–Wiener estimates, $F \in \mathcal{S}(\mathbb{R}^2)$. Thus, $F = \hat{f}$ for some $f \in \mathcal{S}(\mathbb{R}^2)$. It remains to show that f is supported inside the disk D_ϵ . Consider the usual Radon transform $\mathcal{R}f(s, \phi)$ of f . According to the standard Fourier-slice theorem [13, 17, 18, 19, 25, 44], the one-dimensional Fourier transform (denoted by a “hat”) from the variable s to σ gives (up to a fixed constant factor) the values $\widehat{\mathcal{R}f}(\sigma, \psi) = F(\sigma\omega)$ if, as before, $\omega = (\cos \psi, \sin \psi)$. Here \mathcal{R} , as before, denotes the standard Radon transform in the plane. Since functions $F(\sigma\omega)$ of σ , as we have just discussed, are uniformly with respect to ω of a Paley–Wiener class, this implies that $\mathcal{R}f(s, \omega)$ has uniformly with respect to ω bounded support in $|s| < 1 - \epsilon$. Now the “hole theorem” [25, 44] (which is applicable to functions of the Schwartz class), implies that f is supported in D_ϵ .

The last step is to show that $R_S f = g = g_n(r)e^{in\phi}$. This, however, immediately follows from comparing formulas (16) and (7), which finishes the proof of the main lemma, Lemma 7. \square

Let us now return to the proof of Theorem 4. We have proven so far that any partial sum h_n of the Fourier series for g belongs to the range of the operator R_S

acting on smooth functions supported inside the disk D_ϵ . The function g itself is the limit of h_n in $C_0^\infty(S \times (\epsilon, 2 - \epsilon))$. The only thing that remains to be proven is that the range is closed in an appropriate topology. Microlocal analysis can help with this.

Consider R_S as an operator acting from functions defined on the open unit disk D to functions defined on the open cylinder $\Omega = S \times (0, 2)$. As such, it is an FIO [21, 23, 58]. If R_S^t is the dual operator, then $E = R_S^t R_S$ is an elliptic pseudodifferential operator of order -1 [21, Theorem 1], [22].¹

LEMMA 8. *The continuous linear operator $E : H_0^2(D_\epsilon) \mapsto H_{loc}^3(D)$ has zero kernel and closed image.*

Proof. Since $E = R_S^t R_S$, the kernel of this operator coincides with the kernel of R_S acting on $H_0^2(D_\epsilon)$. Since S is closed, it is known that R_S has no compactly supported functions in its kernel [1, 2] (this also follows from analytic ellipticity of E and Theorem 8.5.6 of [27]; see also Lemma 4.4 in [2]). Thus, the statement about the kernel is proven and we only need to prove the closedness of the range.

Let P be a properly supported pseudodifferential parametrix of order 1 for E [63]. Then $PE = I + B$, where B is an infinitely smoothing operator on D . Consider the operator Π that acts as the composition of restriction to D_ϵ and then orthogonal projection onto $H_0^2(D_\epsilon)$ in $H^2(D_\epsilon)$. On $H_0^2(D_\epsilon)$ one has $\Pi PE = I + K$, where K is a compact operator on $H_0^2(D_\epsilon)$. Notice that the operator ΠP is continuous from the Fréchet space $H_{loc}^3(D)$ to $H_0^2(D_\epsilon)$. Due to the Fredholm structure of the operator $\Pi PE = I + K$ acting on $H_0^2(D_\epsilon)$, its kernel is finite-dimensional. Let $M \subset H_0^2(D_\epsilon)$ be a closed subspace of finite codimension complementary to the kernel, so $I + K$ is injective on M and has closed range. Then one can find a bounded operator A in $H_0^2(D_\epsilon)$ such that $A(I + K)$ acts as identity on M . Thus, the operator $A\Pi P$ provides a continuous left inverse to $E : M \mapsto H_{loc}^3(D)$. This shows that the range of E on M is closed in $H_{loc}^3(D)$. On the other hand, the total range of E differs only by a finite dimension from the one on M . Thus, it is also closed. \square

We can now finish the proof of the theorem. Indeed, the last lemma shows that the function $R_S^t g$, being in the closure of the range, is in fact in the range, and thus can be represented as Ef with some $f \in H_0^2(D_\epsilon)$. In other words, $R_S^t(R_S f - g) = 0$. Since the kernel of R_S^t on compactly supported functions is orthogonal to the range of R_S , we conclude that $R_S f - g = 0$. Since $Ef = R_S^t g$ is smooth, due to ellipticity of E we conclude that f is smooth as well. This concludes the proof of the theorem. \square

We would like to finish with some remarks.

- It should be possible to prove that the operator R_S in the situation considered in the text is semi-Fredholm between appropriate Sobolev spaces (analogously to the properties of the standard and attenuated Radon transforms; see, e.g., [24, 44]). This would eliminate the necessity of the closedness of the range discussion in the end of the proof of Theorem 4.

Such a statement could probably be proven either by using FIO techniques, or by controlling dependence on n of the constant C and of the radius of the circle S_0 in Lemma 6. The former approach would be better, being more general.

- Proving compactness of support of function f in Lemma 7, we used the standard Radon transform and the “hole theorem.” Instead, one could probably use the fact that Fourier transform of f is, by construction, a Paley–Wiener class CR-function on the three-dimensional variety of points $\sigma\omega$ in \mathbb{C}^2 and then use an appropriate mandatory analytic extension theorem in the spirit of [54].

¹Bolker’s injective immersion condition [21, 22], which is needed for validity of this result, is satisfied here, as shown in the proof of Lemma 4.3 in [2].

- We considered the situation most natural for tomographic imaging, when the functions to reconstruct are supported inside the aperture curve S . What happens when the supports of functions extend outside the circle S ? It is known that compactly supported functions [2] (or even those belonging to L_p with sufficiently small p [1]) can still be uniquely reconstructed. Necessity of the range conditions we derived apparently still holds, and they are still sufficient for finite Fourier series. However, many things do go wrong in this case. Our proof of the closedness of the range fails (in particular, since Bolker's condition for the corresponding FIO does not hold anymore, which was also the main hurdle in proving the results of [2]). Moreover, the range will no longer be closed. Indeed, reconstruction will become unstable, since due to standard microlocal reasons [32, 35, 40, 61, 70], some parts of the wave front set of the function outside S will not be stably recoverable. This means, in particular, that nonsmooth functions can have smooth circular Radon images. This, in turn implies that the range is not closed in the spaces under consideration, and so sufficiency of the range conditions should fail. We are not sure what kind of range description, if any, could work in this situation. By the way, the nice backprojection-type inversion formulas available in odd dimensions [16] also fail for such functions.
- It would be interesting to understand range conditions in the case of a closed curve S different from a circle. Since our method uses rotational invariance, it is not directly applicable to this situation.
- Our result is stated and proven in two dimensions only. It is possible that a similar approach might work in higher dimensions. As we have been notified by D. Finch, he and Rakesh have recently obtained by different methods some range descriptions in three dimensions [15].

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