Table of Integrals

ELEMENTARY FORMS

1. $\int u\,dv = uv - \int v\,du$

2. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C \quad \text{if } n \neq -1$

3. $\int \frac{du}{u} = \ln|u| + C$

4. $\int e^u\,du = e^u + C$

5. $\int a^u\,du = \frac{a^u}{\ln a} + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

TRIGONOMETRIC FORMS

19. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

20. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

21. $\int \tan^2 u\,du = \tan u - u + C$

22. $\int \cot^2 u\,du = -\cot u - u + C$

23. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

24. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

25. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

26. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

27. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

28. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

29. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C \quad \text{if } a^2 \neq b^2$

(Continued on Rear Endpaper)

Table of Integrals (cont.)

30. $\displaystyle\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C \quad \text{if } a^2 \neq b^2$

31. $\displaystyle\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C \quad \text{if } a^2 \neq b^2$

32. $\displaystyle\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

33. $\displaystyle\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

34. $\displaystyle\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du \quad \text{if } n \neq 1$

35. $\displaystyle\int \cot^n u\, du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du \quad \text{if } n \neq 1$

36. $\displaystyle\int \sec^n u\, du = \frac{1}{n-1}\sec^{n-2} u \tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du \quad \text{if } n \neq 1$

37. $\displaystyle\int \csc^n u\, du = -\frac{1}{n-1}\csc^{n-2} u \cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du \quad \text{if } n \neq 1$

38. $\displaystyle\int u \sin u\, du = \sin u - u\cos u + C$

39. $\displaystyle\int u \cos u\, du = \cos u + u\sin u + C$

40. $\displaystyle\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1}\cos u\, du$

41. $\displaystyle\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du$

FORMS INVOLVING $\sqrt{u^2 \pm a^2}$

42. $\displaystyle\int \sqrt{u^2 \pm a^2}\, du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

43. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

FORMS INVOLVING $\sqrt{a^2 - u^2}$

44. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

45. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

DEFINITION 4.2.1

Let V be a nonempty set (whose elements are called vectors) on which are defined an addition operation and a scalar multiplication operation with scalars in F. We call V a **vector space over** F, provided the following ten conditions are satisfied:

A1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, the sum $\mathbf{u} + \mathbf{v}$ is also in V. We say that V is **closed under addition.**

A2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ in V and each scalar k in F, the scalar multiple $k\mathbf{v}$ is also in V. We say that V is **closed under scalar multiplication**.

A3. *Commutativity of addition:* For all $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

A4. *Associativity of addition:* For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A5. *Existence of a zero vector in V:* In V there is a vector, denoted $\mathbf{0}$, satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \qquad \text{for all } \mathbf{v} \in V.$$

A6. *Existence of additive inverses in V:* For each vector $\mathbf{v}$ In V, there is a vector, denoted $-\mathbf{v}$, in V such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

A7. *Unit property:* For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

A8. *Associativity of scalar multiplication:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(rs)\mathbf{v} = r(s\mathbf{v}).$$

A9. *Distributive property of scalar multiplication over vector addition:* For all $\mathbf{u}$, $\mathbf{v} \in V$ and all scalars $r \in F$,

$$r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}.$$

A10. *Distributive property of scalar multiplication over scalar addition:* For all $\mathbf{v} \in V$ and all scalars $r, s \in F$,

$$(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}.$$

Theorem 6.2.5

Consider the differential equation

$$P(D)y = 0. \tag{6.2.6}$$

Let $r_1, r_2, \ldots, r_k$ be the distinct roots of the auxiliary equation, so that

$$P(r) = (r - r_1)^{m_1}(r - r_2)^{m_2} \cdots (r - r_k)^{m_k},$$

where m_i denotes the multiplicity of the root $r = r_i$.

1. If r_i is *real*, then the functions $e^{r_i x}, xe^{r_i x}, \ldots, x^{m_i - 1}e^{r_i x}$ are linearly independent solutions to Equation (6.2.6) on any interval.
2. If r_j is *complex*, say $r_j = a + ib$ (a and b are real, with $b \neq 0$), then the functions

$$e^{ax}\cos bx, xe^{ax}\cos bx, \ldots, x^{m_j-1}e^{ax}\cos bx$$

$$e^{ax}\sin bx, xe^{ax}\sin bx, \ldots, x^{m_j-1}e^{ax}\sin bx$$

corresponding to the conjugate roots $r = a \pm ib$ are linearly independent solutions to Equation (6.2.6) on any interval.
3. The n real-valued solutions $y_1, y_2, \ldots, y_n$ to Equation (6.2.6) that are obtained by considering the distinct roots $r_1, r_2, \ldots, r_k$ are linearly independent on any interval. Consequently, the general solution to Equation (6.2.6) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

6.3 The Method of Undetermined Coefficients

$F(x)$		**Usual Trial Solution**
$cx^k e^{ax}$	If $P(a) \neq 0$:	$y_p(x) = e^{ax}(A_0 + \cdots + A_k x^k)$
$cx^k e^{ax}\cos bx$ or $cx^k e^{ax}\sin bx$	If $P(a+ib) \neq 0$:	$y_p(x) = e^{ax}[A_0 \cos bx + B_0 \sin bx$ $+x(A_1 \cos bx + B_1 \sin bx)$ $+ \cdots +$ $x^k(A_k \cos bx + B_k \sin bx)]$

$F(x)$		**Modified Trial Solution**
$cx^k e^{ax}$	If a is a root of $P(r)$ of multiplicity m:	$y_p(x) = x^m e^{ax}(A_0 + \cdots + A_k x^k)$
$cx^k e^{ax} \cos bx$ or $cx^k e^{ax} \sin bx$	If $a + ib$ is a root of $P(r)$ of multiplicity m:	$y_p(x) = x^m e^{ax}[A_0 \cos bx + B_0 \sin bx + x(A_1 \cos bx + B_1 \sin bx) + \cdots + x^k(A_k \cos bx + B_k \sin bx)]$

Theorem 7.4.8

Let A be an $n \times n$ matrix of real constants.

1. Suppose λ is a real eigenvalue of A with corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then k linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x}_j(t) = e^{\lambda t}\mathbf{v}_j, \qquad j = 1, 2, \ldots, k.$$

2. Suppose $\lambda = a + ib$ is a complex eigenvalue of A with corresponding linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, where $\mathbf{v}_j = \mathbf{r}_j + i\mathbf{s}_j$. Then k complex-valued solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{u}_j(t) = e^{\lambda t}\mathbf{v}_j, \qquad j = 1, 2, \ldots, k$$

and $2k$ *real-valued* linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\begin{aligned}
\mathbf{x}_{11}(t) &= e^{at}(\cos bt\, \mathbf{r}_1 - \sin bt\, \mathbf{s}_1), & \mathbf{x}_{12}(t) &= e^{at}(\sin bt\, \mathbf{r}_1 + \cos bt\, \mathbf{s}_1),\\
\mathbf{x}_{21}(t) &= e^{at}(\cos bt\, \mathbf{r}_2 - \sin bt\, \mathbf{s}_2), & \mathbf{x}_{22}(t) &= e^{at}(\sin bt\, \mathbf{r}_2 + \cos bt\, \mathbf{s}_2),\\
&\vdots & &\vdots\\
\mathbf{x}_{k1}(t) &= e^{at}(\cos bt\, \mathbf{r}_k - \sin bt\, \mathbf{s}_k), & \mathbf{x}_{k2}(t) &= e^{at}(\sin bt\, \mathbf{r}_k + \cos bt\, \mathbf{s}_k).
\end{aligned}$$

Further, the set of all solutions to $\mathbf{x}' = A\mathbf{x}$ obtained in this manner is linearly independent on any interval.